Chapter 11 – FT properties and Nyquist sampling criteria

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Quick Recap:

Imaging Equation

1D: $$s(k_x) = \int dx \cdot \rho(x) \cdot e^{-i \cdot 2 \cdot \pi \cdot k \cdot x}$$

$$x \to \vec{r} = (x, y, z)$$

3D: $$s(k) = \int d^3 r \cdot \rho(r) \cdot e^{-i 2\pi (\vec{r} \cdot \vec{k})}$$

1) All dimensions are equivalent
2) k along each direction is determined by the gradient in that direction

$$s(k_x, k_y, k_z) = \iiint dx \cdot dy \cdot dz \cdot \rho(x, y, z) \cdot e^{-i 2\pi (k_x x + k_y y + k_z z)}$$

$$k_x(t) = \gamma \int^t G_x(t')dt', \quad k_y(t) = \gamma \int^t G_y(t')dt', \quad k_z(t) = \gamma \int^t G_z(t')dt'$$

Quick Recap:

Imaging Equation

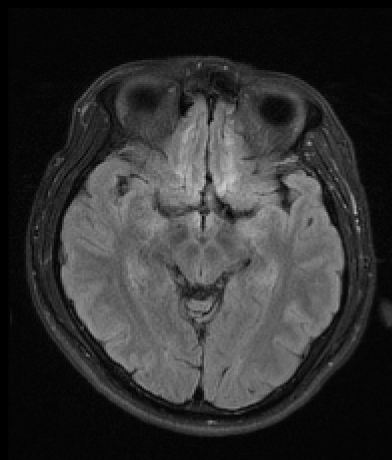
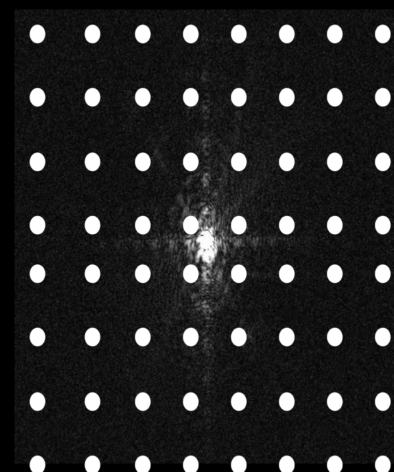


Image space

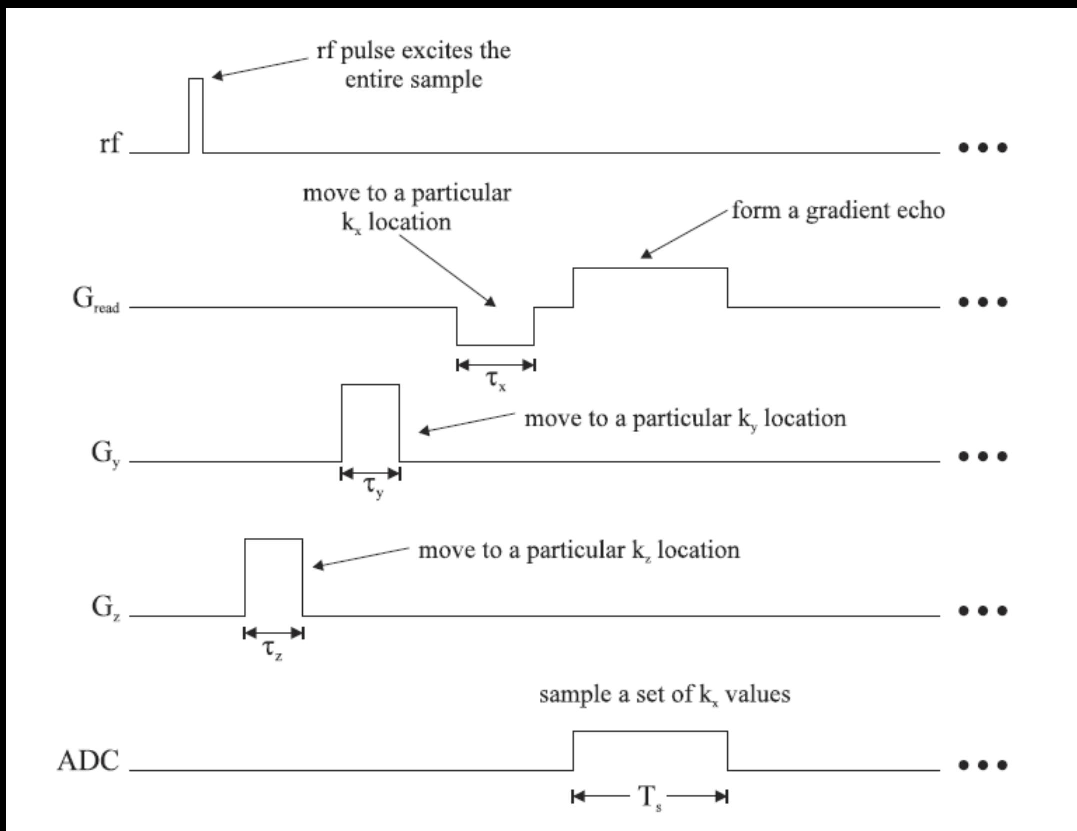


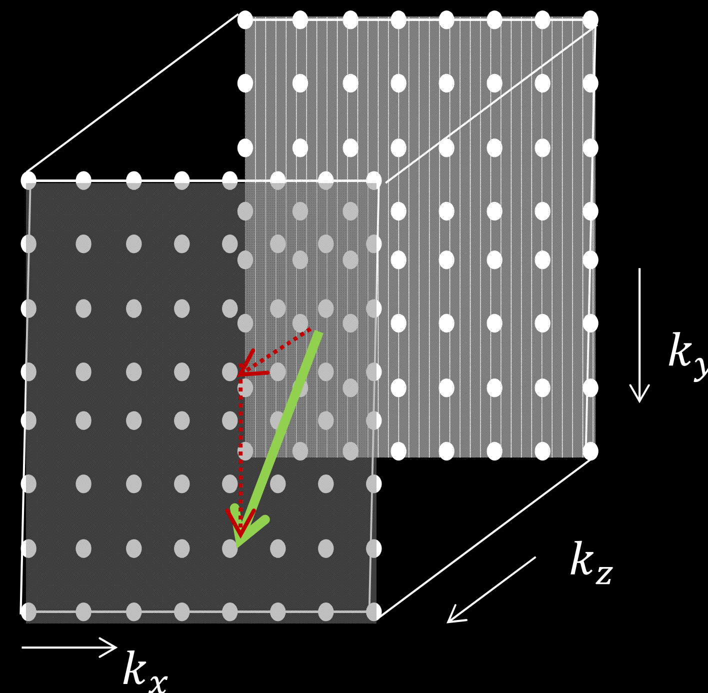
k-space

Filling up the corresponding 2D/3D k-space to obtain the image

Quick Recap:

Multidimensional Spatial Encoding (2D/3D imaging)



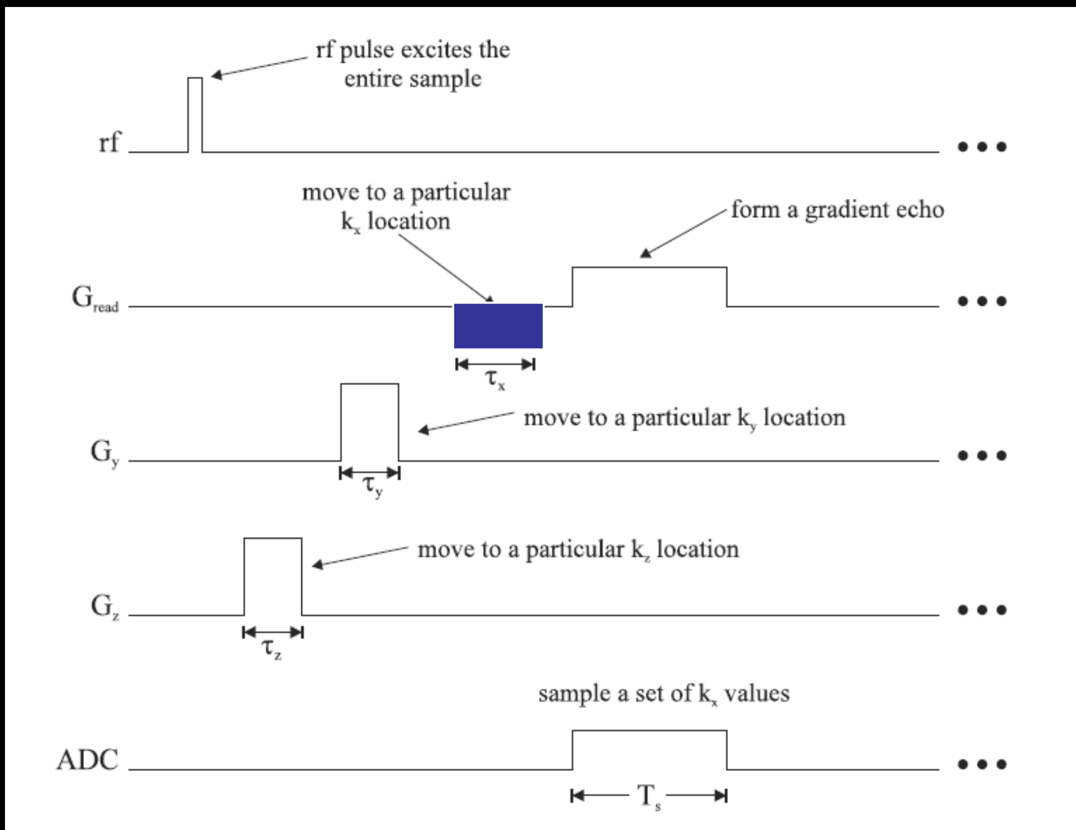


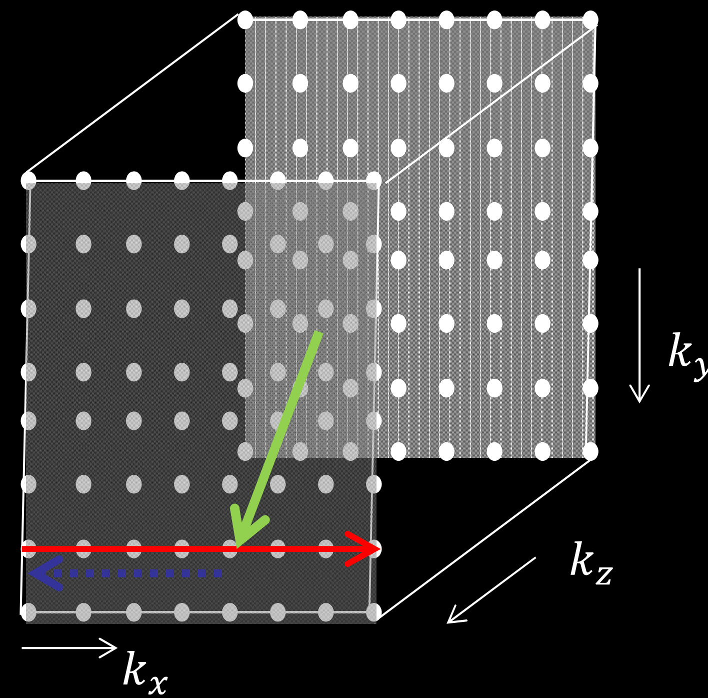
$$k_y = \gamma \cdot G_y \cdot \tau_y$$

$$k_z = \gamma \cdot G_z \cdot \tau_z$$

Quick Recap:

Multidimensional Spatial Encoding (2D/3D imaging)





$$k_y = \bar{\gamma} \cdot G_y \cdot \tau_y$$

$$\Delta k_y = \bar{\gamma} \cdot \Delta G_y \cdot \tau_y$$

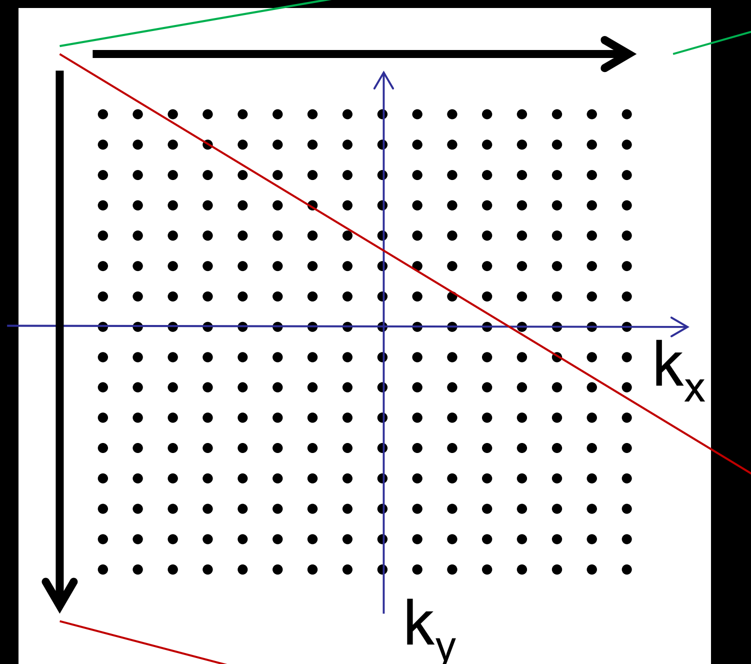
$$\Delta k_z = \bar{\gamma} \cdot \Delta G_z \cdot \tau_z$$

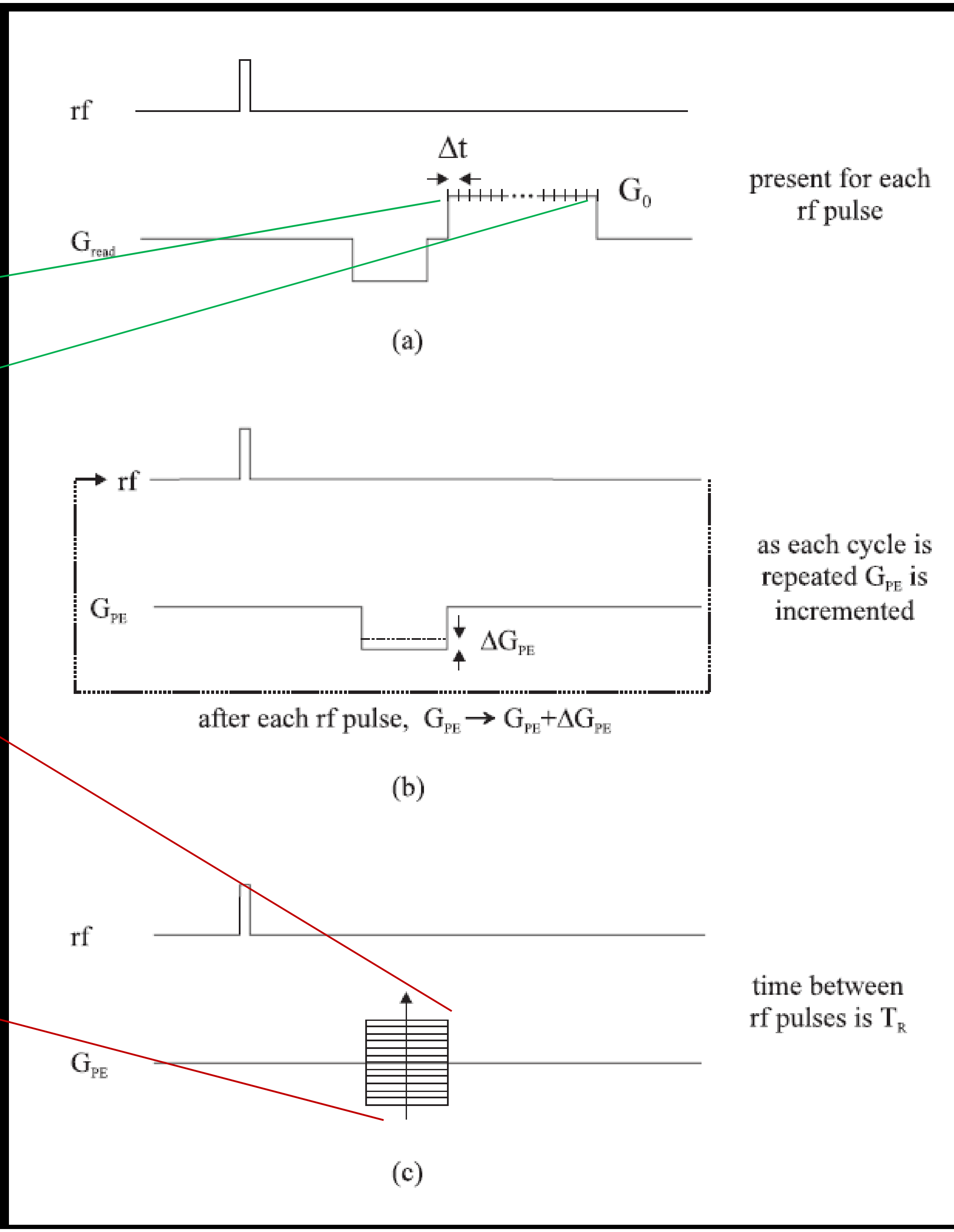
$$\Delta k_x = \bar{\gamma} \cdot G_x \cdot \Delta t$$

$$k_x = \bar{\gamma} \cdot G_x \cdot t \qquad k_z = \bar{\gamma} \cdot G_z \cdot \tau_z$$

Quick Recap:

Phase encoding



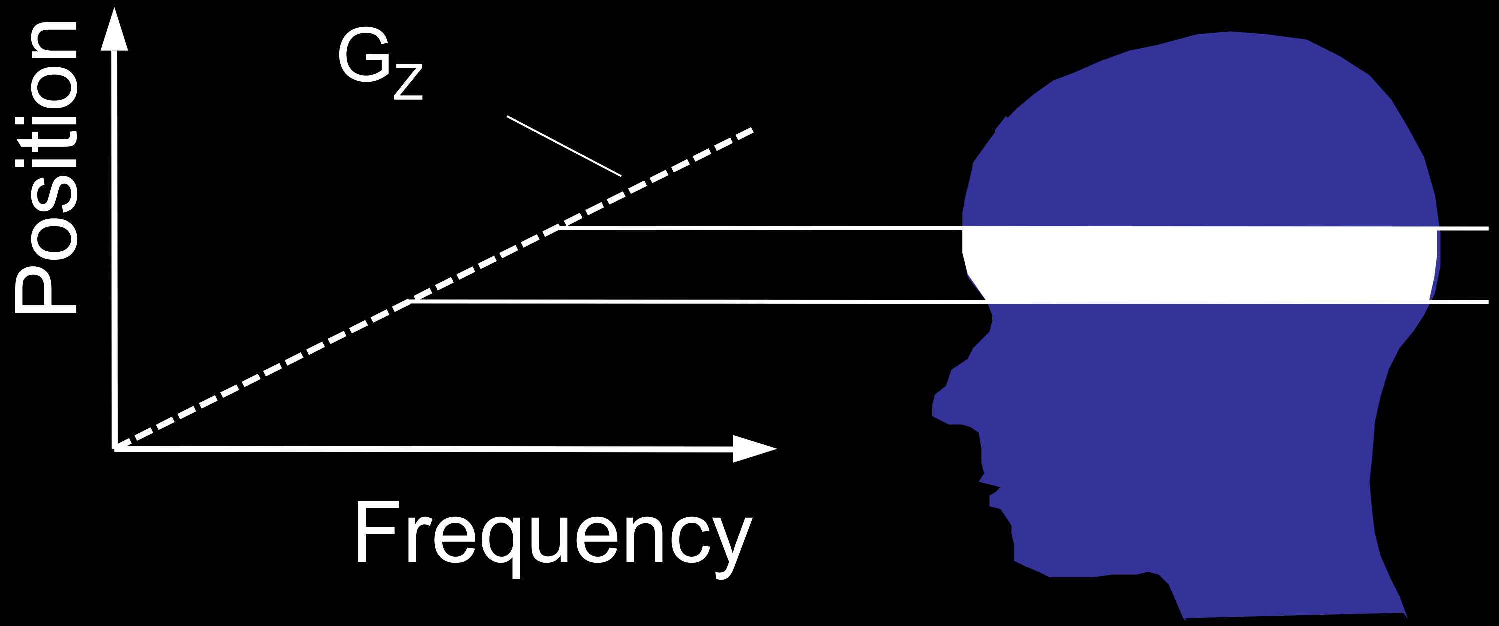


$$k_y = \gamma \cdot G_y \cdot t$$

$$\Delta k_y = \gamma \cdot \Delta G_y \cdot \tau$$

Quick Recap:

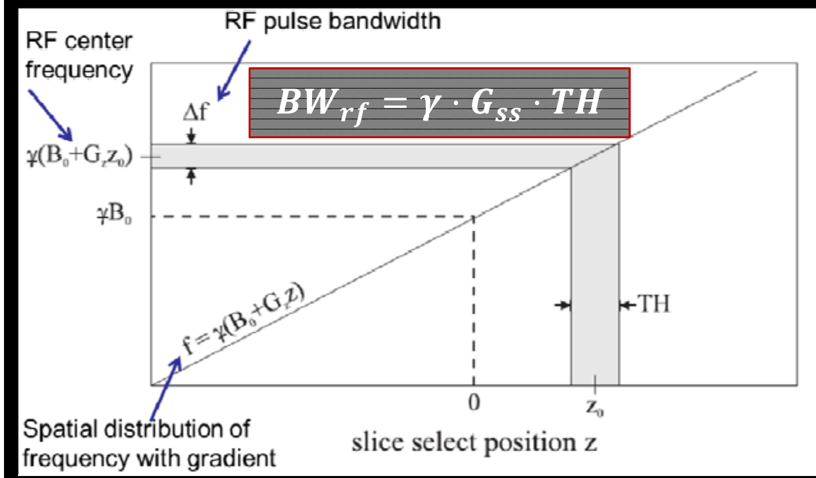
Slice Selection



Quick Recap:

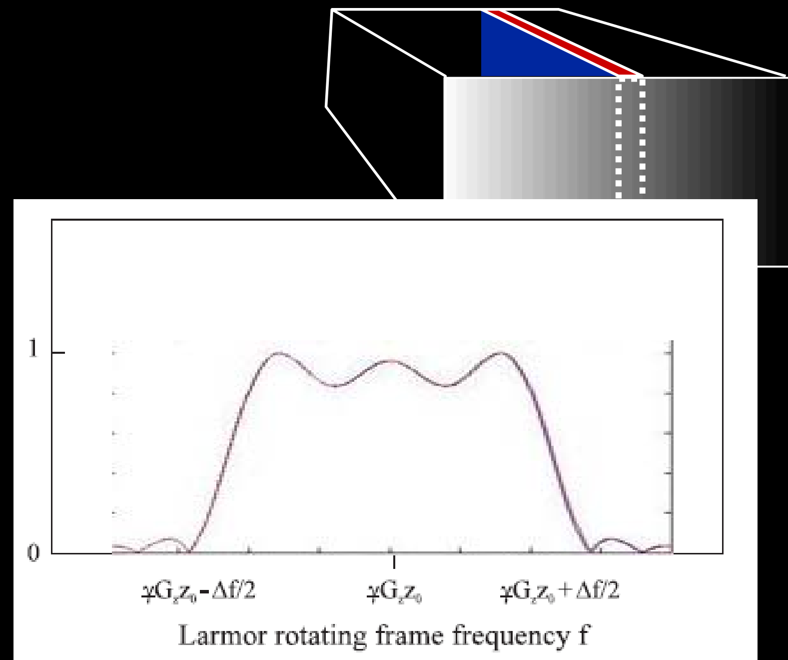
Slice selection

Slice selection

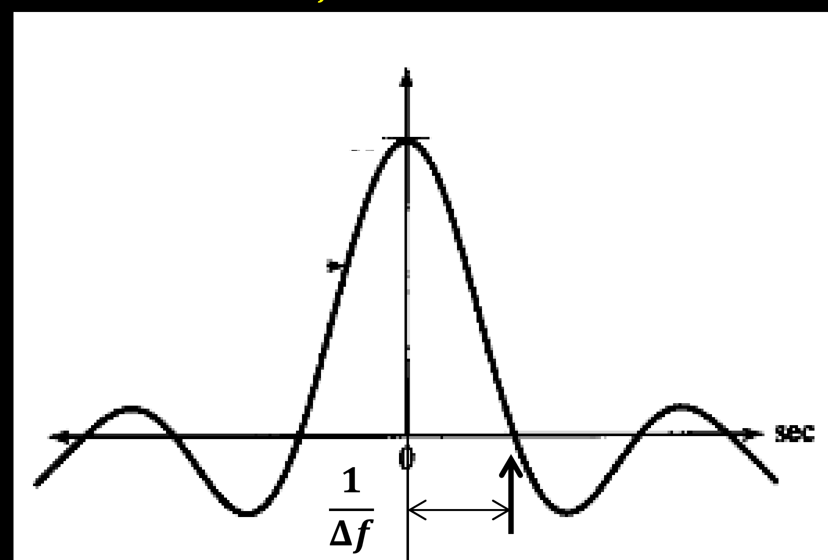


$$\Delta f = BW_{rf}$$

$$\tau_{rf} = n_{zc} \cdot \left(\frac{1}{BW_{rf}}\right)$$

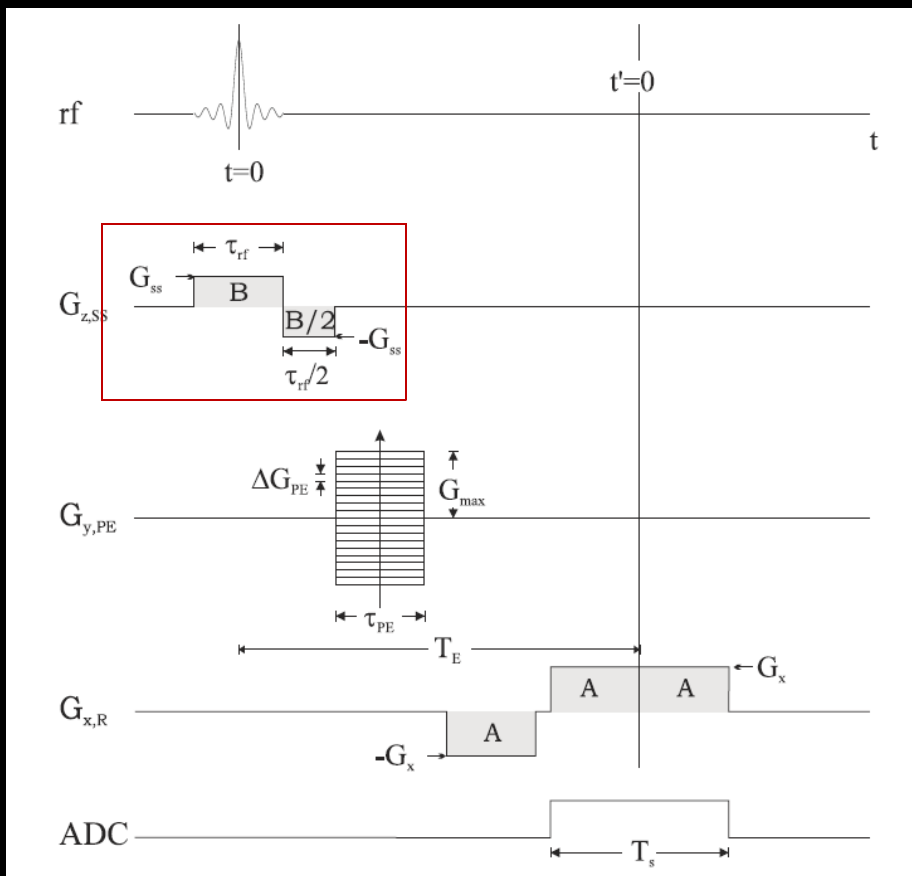


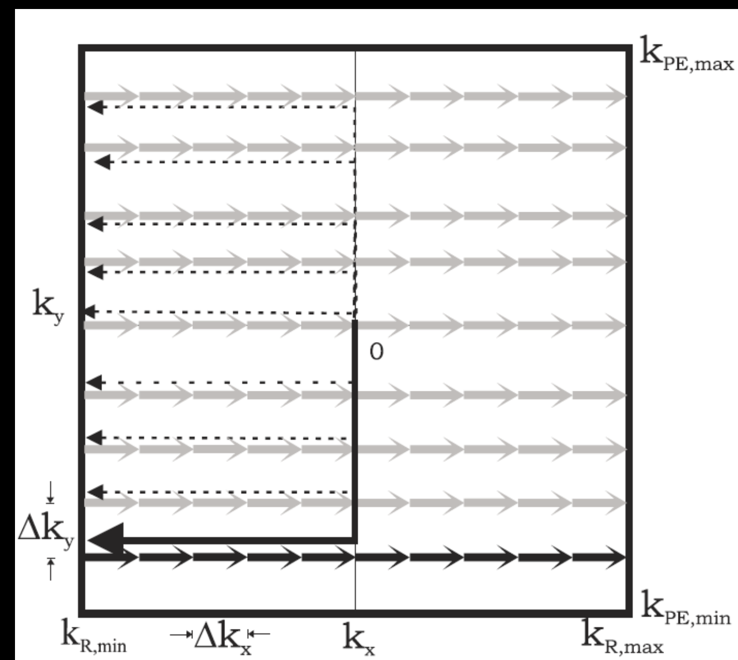
$$BW_{rf} = \Delta f = \gamma \cdot G_{ss} \cdot TH$$



Quick Recap:

2D Imaging sequence



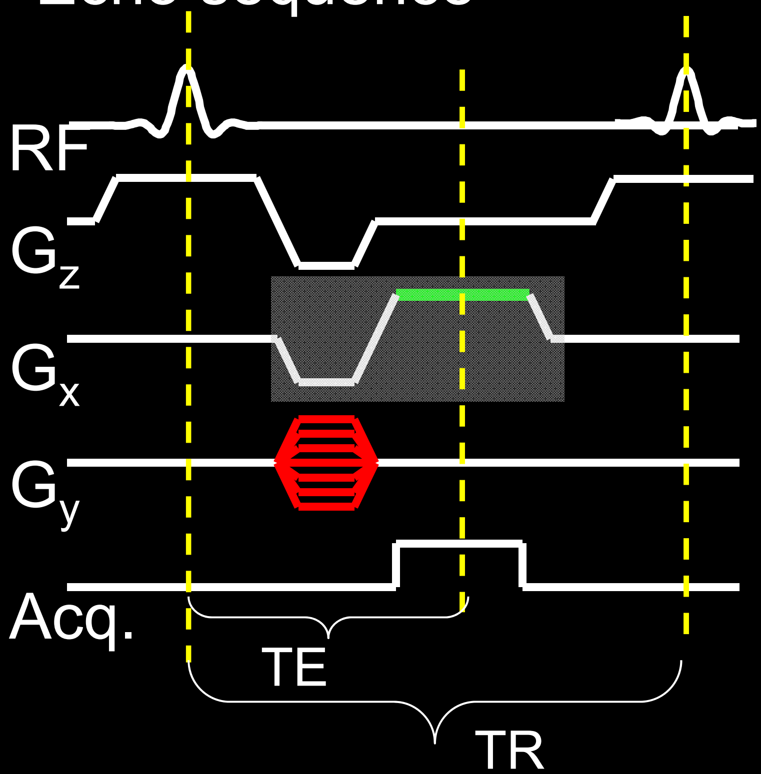


$$k_{read\_min} = \bar{\gamma} \cdot -G_{read} \cdot \frac{T_s}{2}$$

Quick Recap:

2D imaging: acquiring line by line

Typical Gradient Echo sequence



k-space

Image space

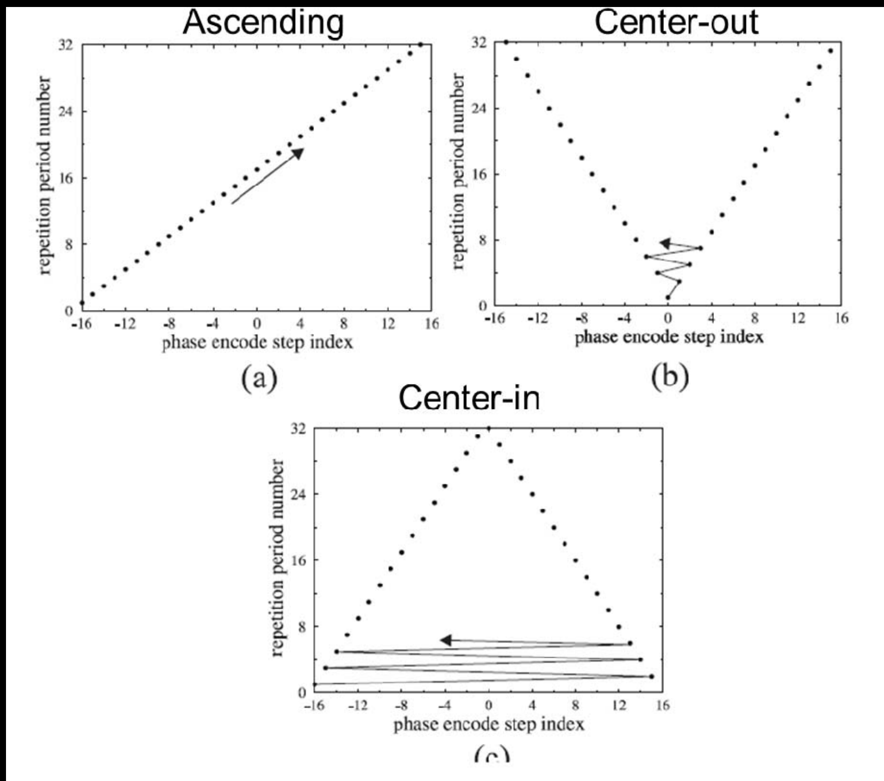
Total imaging time for a single slice – N_y x TR

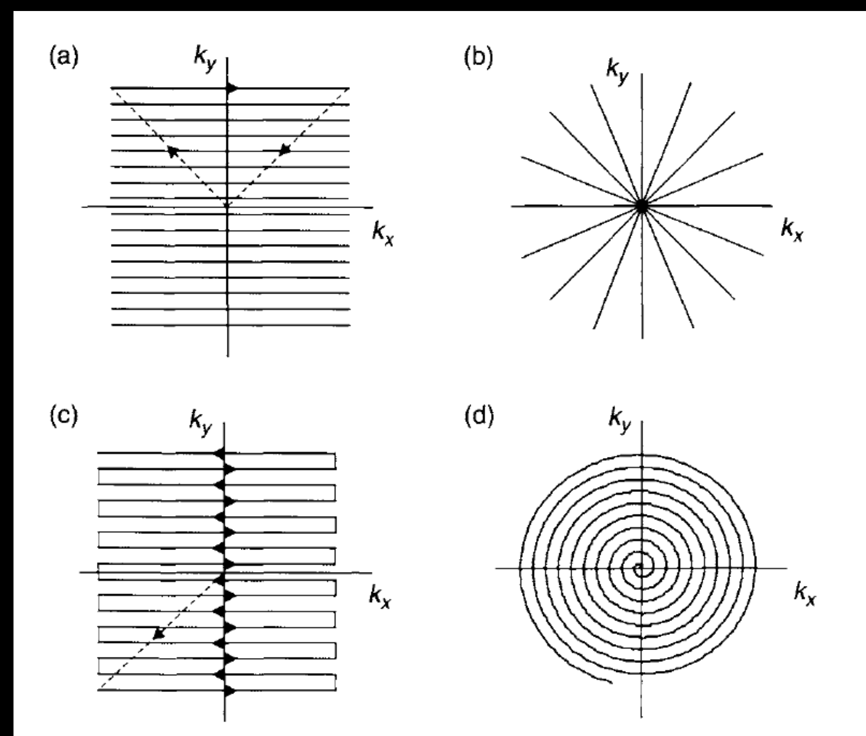
Sampling of the signal line by line forming the image in Spatial frequency domain

Animation Source: Internet

Quick Recap:

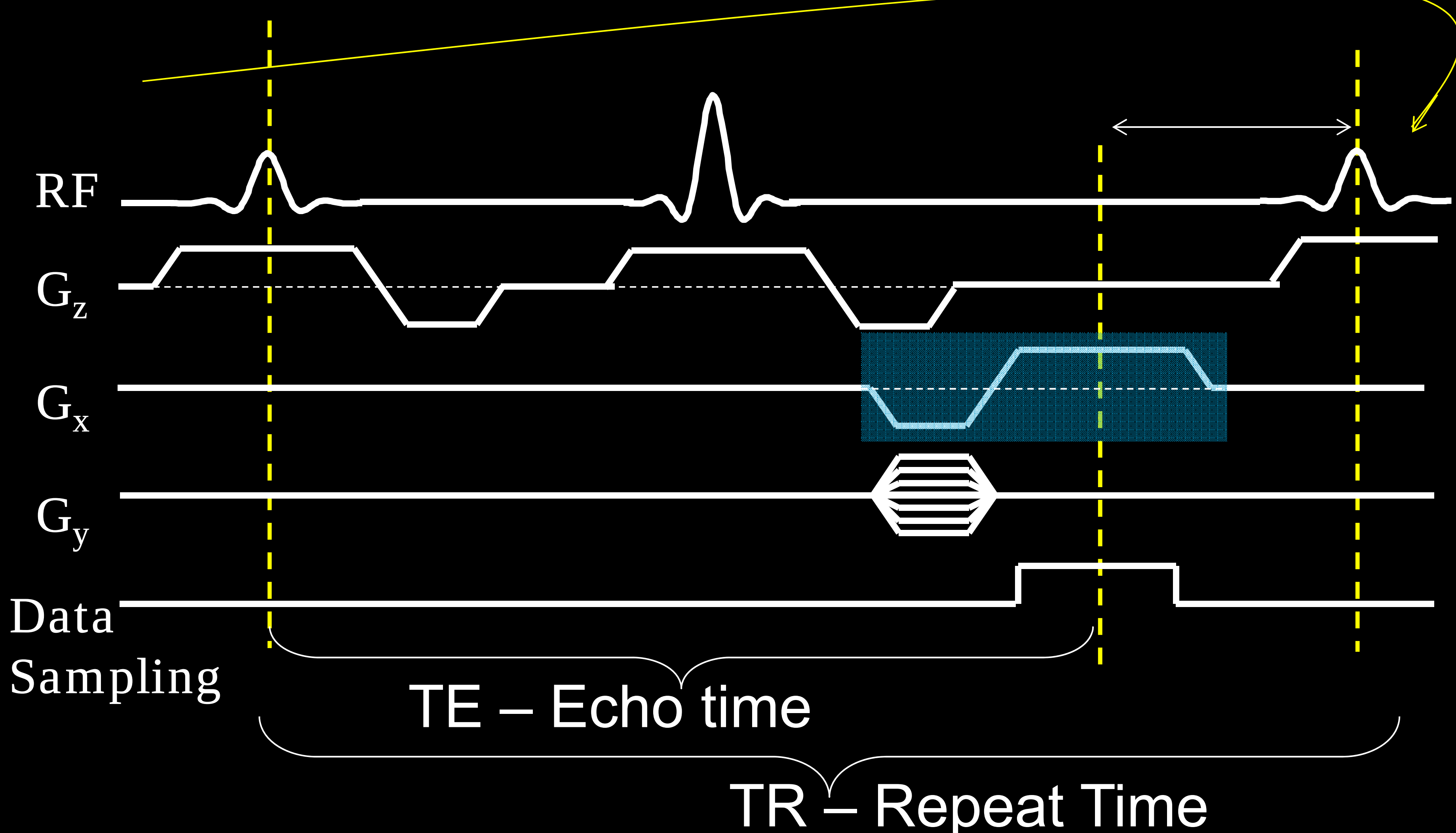
Phase encoding order





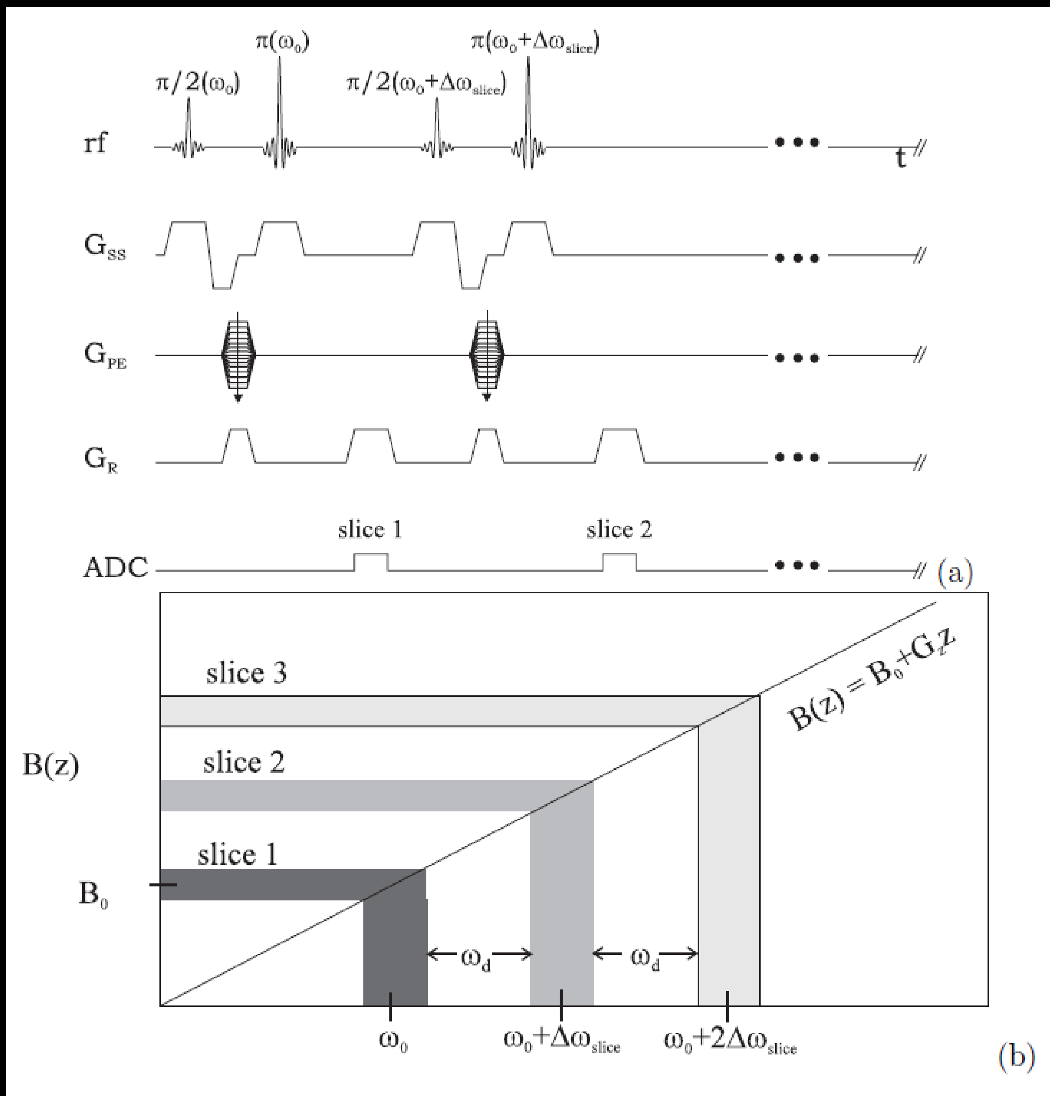
Quick Recap:

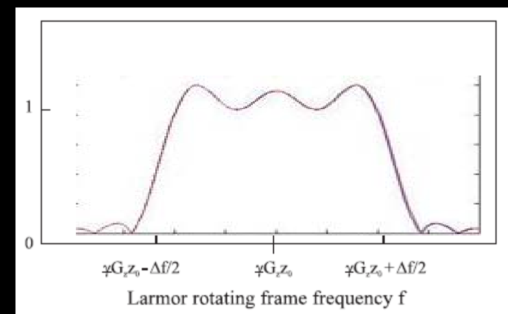
Typical Spin Echo sequence

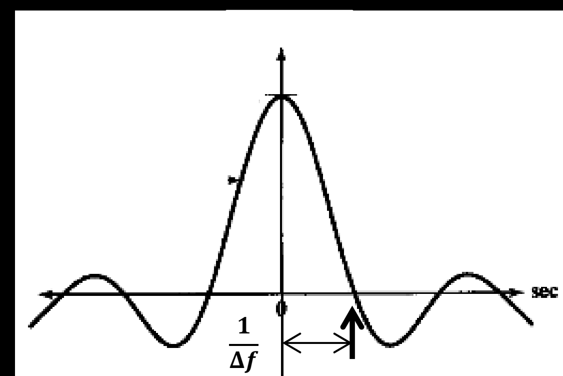


Quick Recap:

Multi-slice 2D imaging







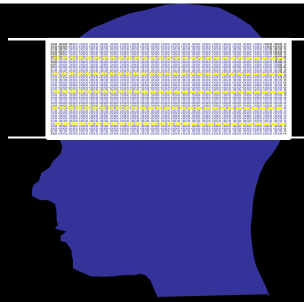
$$\gamma G_{ss}(TH + d) = \Delta\omega_{slice}$$

$$d = \frac{(\Delta\omega_{slice} - 2\pi BW_{rf})}{\gamma G_s}$$

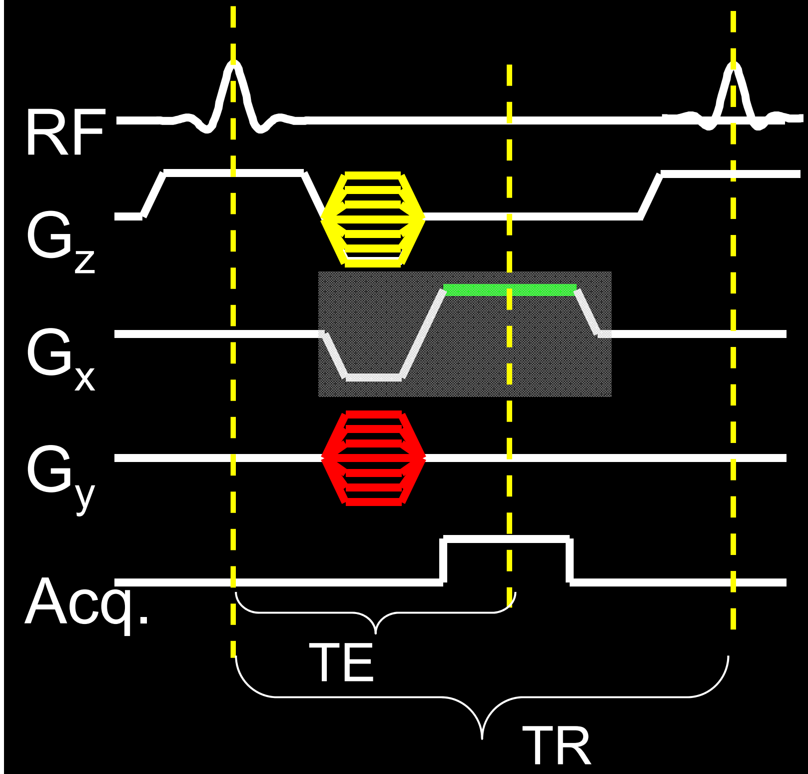
Slice interleaved acquisition - Odd/Even

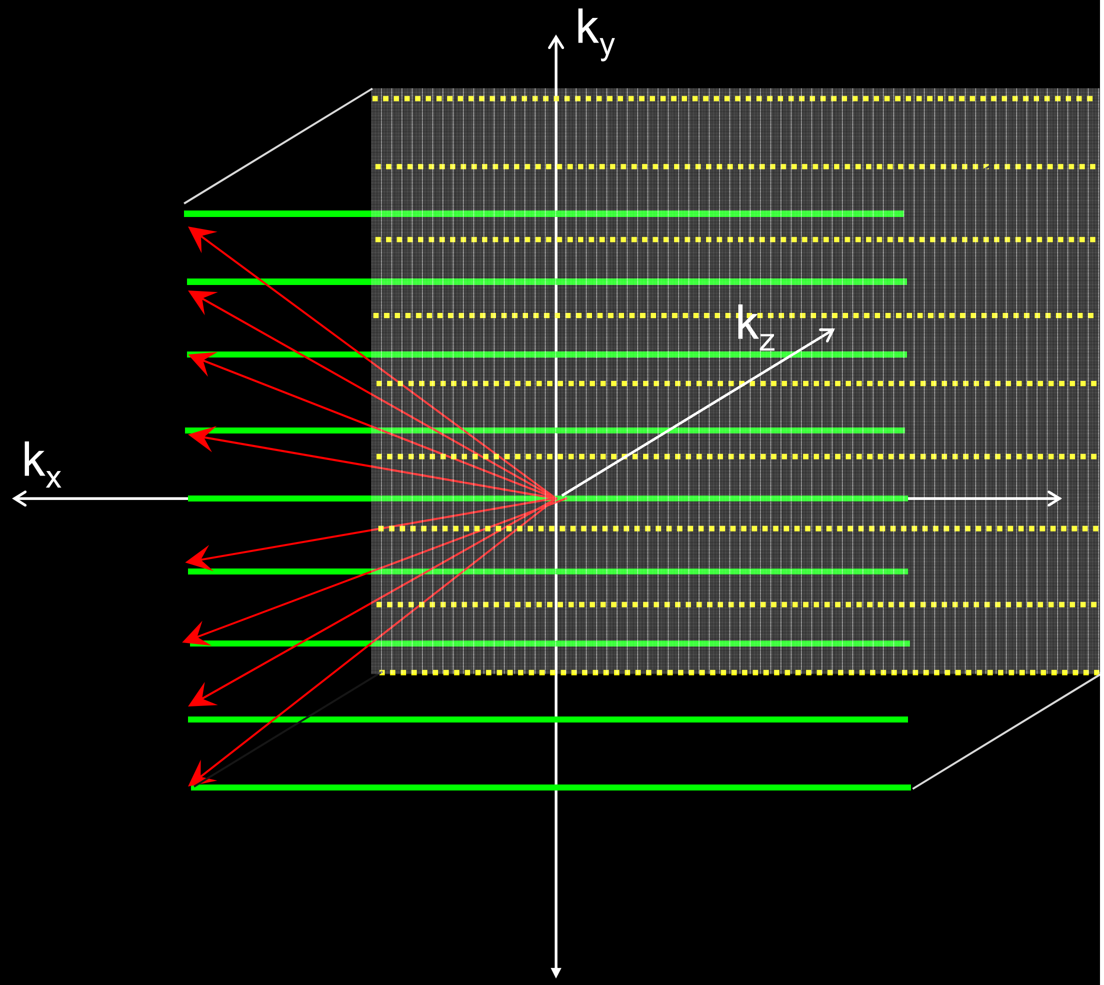
Quick Recap:

3D volumetric imaging



3D GRE - with slab excitation and partition encoding





Total Imaging time – N_z x N_y x TR

Slab Excitation and partition encoding in the z direction.

This class - Key points

- Properties of Fourier Transform
- Sampling of the analog signal
- Discrete Fourier Transform
- Finite Sampling & Sampling rate

Continuous FT Properties

FT is the bridge between two 'conjugate' domains

- Linearity
- Duality
- Spatial Scaling
- Space Shifting (shift theorem)
- Even/odd symmetry
- Convolution
- Derivative
- Parseval (energy)

Continuous FT Properties

- **Linearity:**

$$\mathcal{F}[af(x) \pm bg(x)] = aF(k) \pm bG(k)$$

- **Implications for MRI:**
  - **Linear calculation can be done in either k-space or image domain**

Continuous FT Properties

- **Spatial Scaling:**

$$\mathcal{F}[\boldsymbol{h}(a\boldsymbol{x})]=\mathrm{H}(k/a)\ /|a|$$

- **Implications for MRI:**
  - **Stretching or shrinking in, say spatial domain, correspondingly scales the k-variable**
  - **It also scales the amplitude of the k-space correspondingly**

Continuous FT Properties

- **Shift Theorem:**

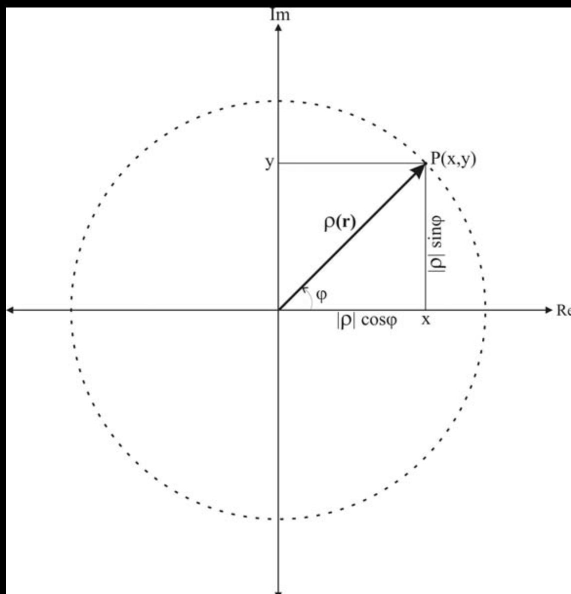
Shift in one domain <=> Additional Linear phase in another

e.g. $s_m(k) \to s_m(k - k_0) \Rightarrow \hat{\rho}(x) \to \hat{\rho}(x)e^{i2\pi k_0 x}$

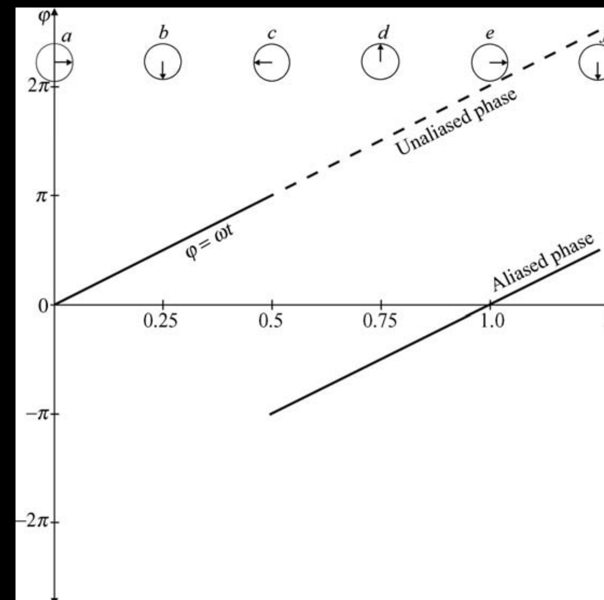
$s_m(k) \to s_m(k)\, e^{-i2\pi k x_0} \Leftarrow \hat{\rho}(x) \to \hat{\rho}(x - x_0)$

- **Implications for MRI:**
  - Motion/displacement of the object will introduce phase error between k-space lines, introducing artifacts (Chap.23)
  - A rigid body motion induced phase error can be recovered by adjusting the phase of corresponding k-space lines
  - Over coverage in the k-space center is necessary to ensure a high signal baseline in the image

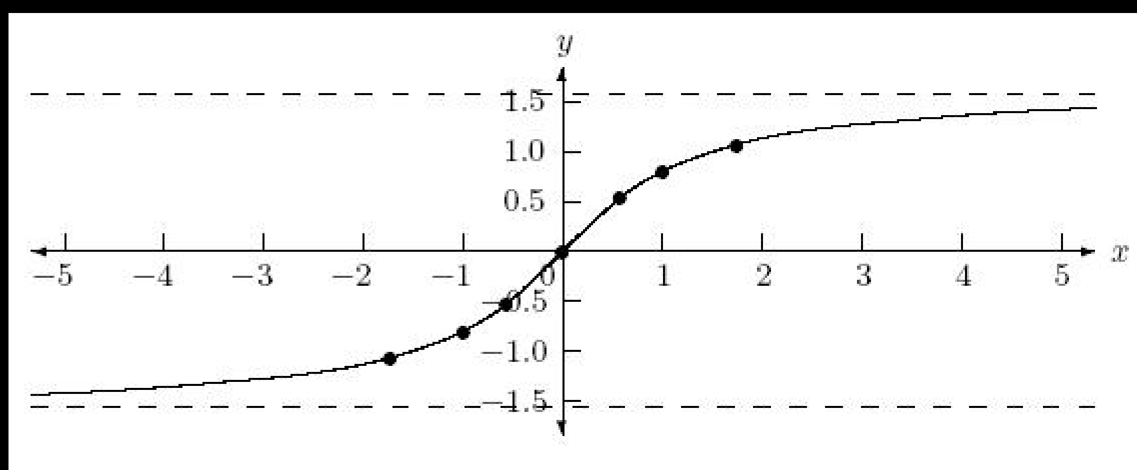
Phase Aliasing



$$\phi = \tan^{-1}\frac{Imaginary}{real}$$

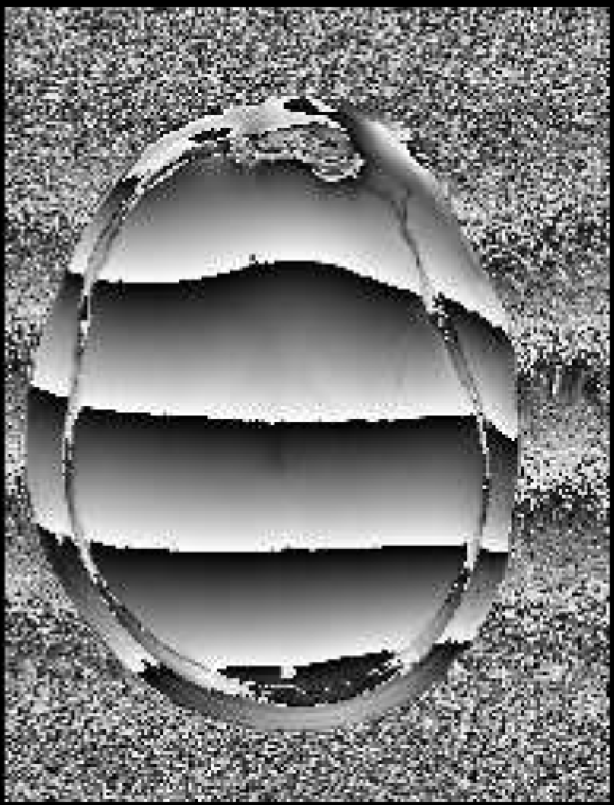


arctan function



Shift in k-space

One point



How many points?

$\hat{\rho}(x)$ -> $\hat{\rho}(x)e^{i2\pi k_0 x}$

$\hat{\rho}(n\Delta x)$ -> $\hat{\rho}(n\Delta x)e^{i2\pi \cdot m\Delta k_0 \cdot n\Delta x}$

Continuous FT Properties

- Convolution theorem

$$\mathcal{F}(\boldsymbol{g}(\boldsymbol{x}) \cdot \boldsymbol{h}(\boldsymbol{x})) = \boldsymbol{G}(\boldsymbol{k}) * \boldsymbol{H}(\boldsymbol{k}) = \int dk' G(k') H(k - k')$$

- Implications for MRI:
  - Filtering effects can be understood either by multiplying the filter in image domain or convoluting with the FT form of the filter in $k-space$
  - Important for understanding finite sampling (sampling + windowing) effects in DFT

Continuous FT Properties

- Conjugate symmetry (partial FT)

$$\mathrm{Real}[\mathcal{F}(f(x))] = \mathrm{Real}\,[\mathcal{F}(-f(x))]$$

$$\mathrm{Imag}[\mathcal{F}(f(x))] = -\,\mathrm{Imag}\,[\mathcal{F}(-f(x))]$$

- Implications for MRI:
  - Partially collected k-space may still contains all the information of the object (w/ conditions)
  - Uncollected k-space can be zero padded, or estimated based on the collected portion
  - Example: Partial Fourier Imaging

More Continuous FT Properties

- **Duality**

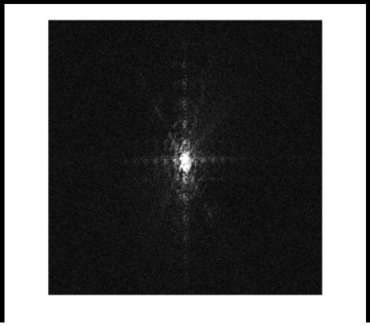
$$h(x) \overset{\mathcal{F}}{\Leftrightarrow} H(k) \qquad \Rightarrow \qquad \mathrm{H}(-x) \overset{\mathcal{F}}{\Leftrightarrow} h(k)$$

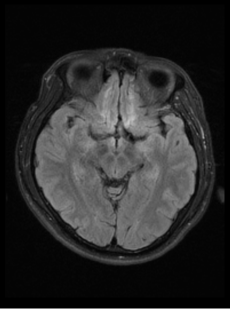
- **Derivative Theorem (e.g. edge finding)**

$$\mathcal{F}(f'(x)) = i2\pi k\,\mathcal{F}(f(x))$$

- **Parseval (constant energy)**

$$\int_{-\infty}^{\infty} dx\,|h(x)|^2 = \int_{-\infty}^{\infty} dk\,|H(k)|^2$$





Important FT pairs

- **Dirac delta function (impulse function)**

$$\delta(k-k_0) = \int_{-\infty}^{\infty} dx\, e^{-i2\pi(k-k_0)x} = \mathcal{F}[e^{i2\pi k_0 x}]$$

$$\delta(x-x_0) = \int_{-\infty}^{\infty} dk\, e^{-i2\pi k(x-x_0)} = \mathcal{F}^{-1}[e^{-i2\pi k x_0}]$$

- **Rectangular Function**

$$\mathcal{F}\left[rect\left(\frac{x}{W}\right)\right] = W sinc(\pi W k)$$

- **Gaussian**

$$\mathcal{F}\left[e^{-\pi x^2}\right] = e^{-\pi k^2}$$

Signal sampling function

$$u(k) = \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) = \Delta k \sum_{p=-\infty}^{\infty} \mathcal{F}[e^{i2\pi p \Delta k x}]$$

$$\Downarrow$$

$$U(x) = \mathcal{F}^{-1}\,[u(k)] = \Delta k \sum_{p=-\infty}^{\infty} e^{i2\pi p \Delta k x}$$

$$\Downarrow$$

$$\sum_{n=-\infty}^{\infty} e^{i2\pi n a} = \sum_{m=-\infty}^{\infty} \delta(a - m)$$

$$\delta(ax) = \delta(x)/|a|$$

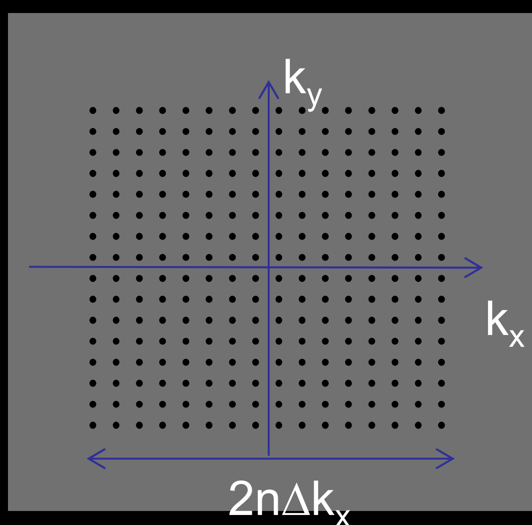
$$U(x) = \sum_{q=-\infty}^{\infty} \delta(x - \frac{q}{\Delta k})$$

Discrete FT (DFT)

- **Approximation of Continuous FT**
- **More practical for MRI**

$$G(p\Delta k) = \sum_{q=-n}^{n-1} g(q\Delta x) e^{-i2\pi q \Delta x p \Delta k}$$

where $\Delta k = 1/L$ and $L = 2n\Delta x$



DFT properties

- Linearity
- Spatial Scaling
- Space Shifting (shift theorem)
- Even/odd symmetry
- Convolution
- Derivative
- Parseval

Sampling, Aliasing and Image Reconstruction

- MRI signal is the finite, truncated, digitized version of the continuous signal picked up by the receive coil

- Discrete sampling is done using ADC (analog-digital converter)

- The sampling must meet certain criteria in order to faithfully represent the original continuous signal

- Failure in meeting the criteria will lead to artifacts

- Physical spin density $\rho(x)$ of the object is real, while reconstructed $\hat{\rho}(x)$ is practically complex

Infinite sampling

- **Sampling function:**

$$u(k) = \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k)$$

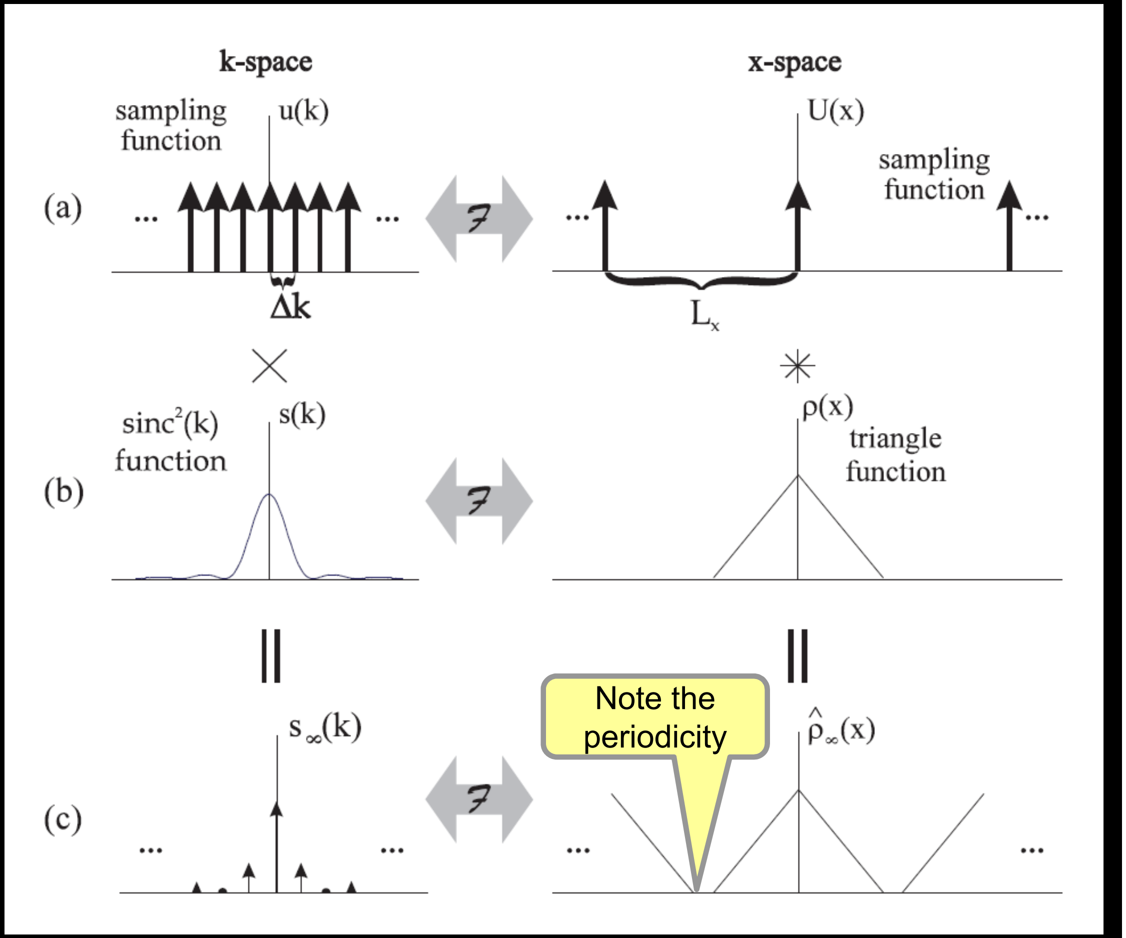
$$U(x) = \sum_{q=-\infty}^{\infty} \delta(x - q/\Delta k)$$

- **Sampled signal:**

$$s_\infty(k) \equiv s(k)u(k)$$
$$= \Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k)\delta(k - p\Delta k)$$

- **Spin density after DFT**

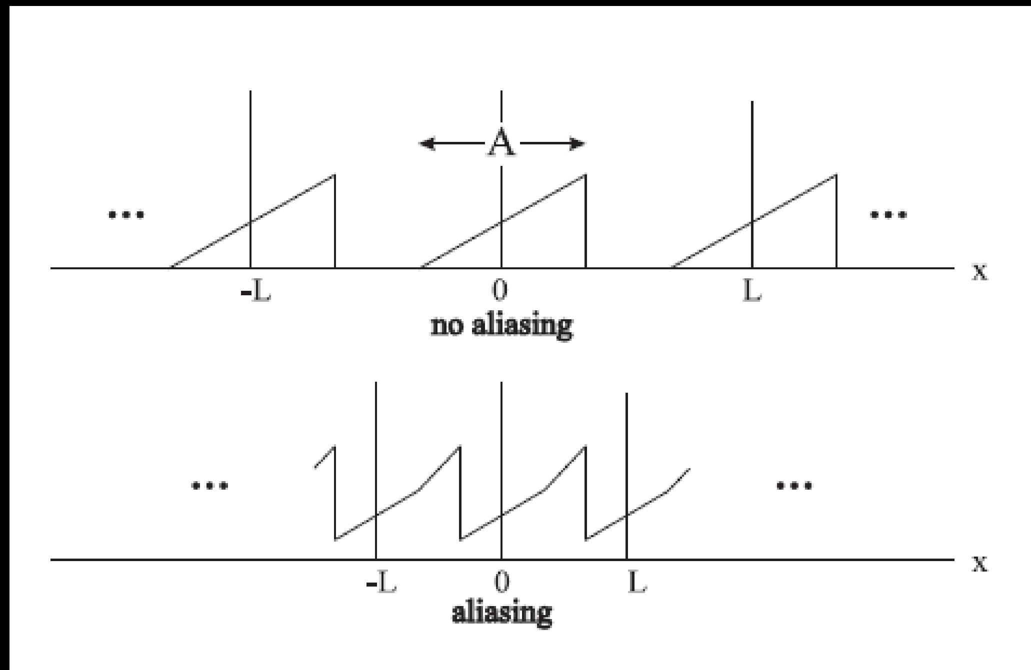
$$\hat{\rho}_\infty(x) = \rho(x) * U(x)$$
$$= \sum_{q=-\infty}^{\infty} \rho(x - q/\Delta k)$$

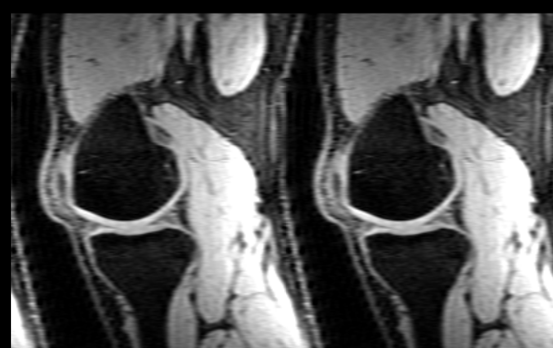


Aliasing and Nyquist Sampling Criterion

- $1/\Delta k = L$
- If object length (A) >L, then overlaps in $\hat{\rho}_{\infty}(x)$ will take place, introducing aliasing artifacts
- To avoid aliasing:
  A<L or $\Delta k<1/A$

Nyquist Sampling Criterion for MRI





Nyquist in Read direction

$$\left.\begin{array}{c} \Delta k_R = \gamma G_R \Delta t \\ \Delta k_R = 1/L_R \\ \Delta t = T_s/N = 1/BW_R \end{array}\right\} \; BW_R = \gamma G_R L_R > \gamma G_R A_R$$

What we usually do is use fixed T_s and GR
but 2 x oversampling, i.e. $\Delta t/2$,along Read direction

$$BW_R = \gamma G_R L_R = \frac{N}{T_s}$$

Nyquist in Phase Encoding direction

$$\left.\begin{array}{c}\Delta k_{PE} = \gamma \Delta G_{PE} \tau_{PE} \\ \Delta k_{PE} = 1/L_{PE} \\ \Delta t = TR = 1/BW_{PE}\end{array}\right\} \quad \gamma \Delta G_{PE} \tau_{PE} = \frac{1}{L_{PE}} < \frac{1}{A_{PE}}$$

What we usually do in MRI is use smaller $\Delta G_{PE}\tau_{PE}$

Corresponding to $G_x \cdot \Delta t$ along the read-direction

Simply collecting more points along PE direction using fixed $\Delta G_{PE}\tau_{PE}$:

1) Will not change L_{PE}

2) But increase the PE resolution

$$\Delta L_{PE} = \frac{L_{PE}}{\mathrm{N}} = \frac{1}{\mathrm{N}\Delta k_{PE}}$$

Finite, discrete sampling

- **Data truncation**

$$s_{m(k)} = s(k) \cdot u(k) \cdot rect(\frac{k+\Delta k/2}{W})$$

Sinc shaped low-pass filter

$$\hat{\rho}(x) = \rho(x) * U(x) * {\color{red} W sinc(\pi W x) e^{-i\pi x \Delta k}}$$

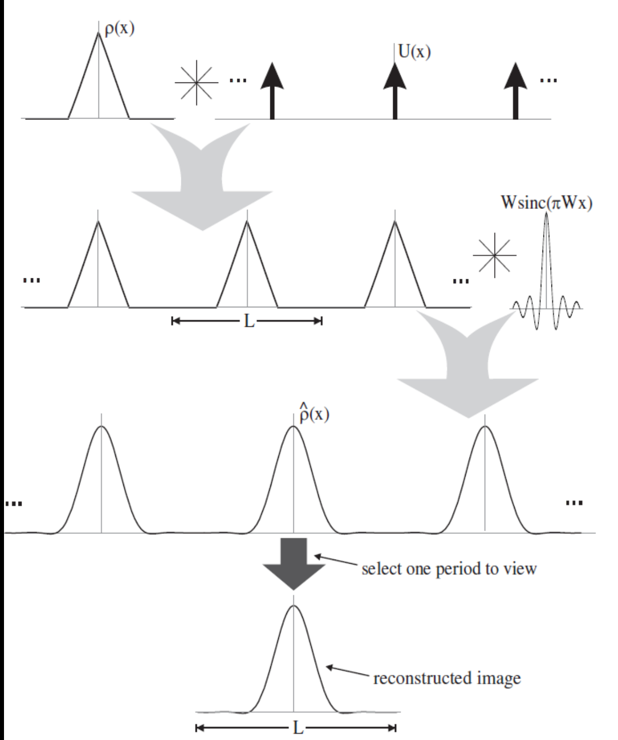
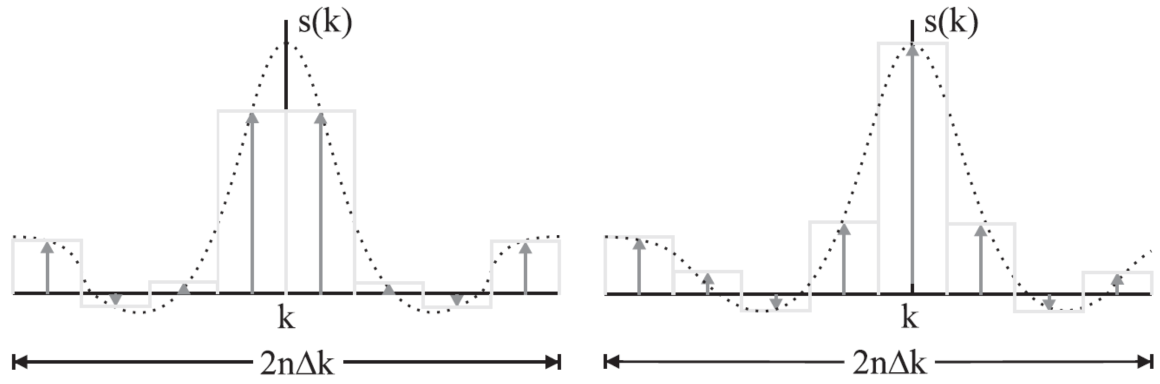


Fig. 12.6

Finite sampling and k-space symmetry

- The question: why $rect(\frac{k+\Delta k/2}{W})$ instead of $rect(\frac{k}{W})$?
- Symmetric (a) vs. MRI standard (b)
  - K-space center (i.e. k=0) and edge not well defined vs. well defined
  - For large 2n, image magnitude will be almost the same
  - A phase shift difference in image phase



(a) (b)

Discrete Fourier Transform(DFT)

$$s(p\Delta k) = \Delta x \sum_{q=-n}^{n-1} \hat{\rho}(q\Delta x) e^{-i\pi pq/n}$$

$$\hat{\rho}(q\Delta x) = \Delta k \sum_{q=-n}^{n-1} s(p\Delta k) e^{i\pi pq/n}$$

- **Resolution (in voxel size)**

$$\Delta L_{RO} = \frac{BW_R}{N_R \gamma G_R}$$

$$\Delta L_{PE} = \frac{1}{N_{PE} \gamma \Delta G_{PE} \tau_{PE}}$$

Homework

- Prob *11.1 - 11.3, 11.8, 12.2, 12.7, 12.8*

Next Session

Chapter 13.1-13.3