Chapter 13 – Filtering effects

Jaladhar Neelavalli, Ph.D.
Assistant Professor,
WSU SOM - Dept. of Radiology

Continuous FT Properties

FT is the bridge between two 'conjugate' domains

- **Linearity**
- **Duality**
- **Spatial Scaling**
- **Space Shifting (shift theorem)**
- **Even/odd symmetry**
- **Convolution**
- **Derivative**
- **Parseval (energy)**

Continuous FT Properties

- **Convolution theorem**

$$\mathcal{F}(\boldsymbol{g}(\boldsymbol{x}) \cdot \boldsymbol{h}(\boldsymbol{x})) = \boldsymbol{G}(\boldsymbol{k}) * \boldsymbol{H}(\boldsymbol{k}) = \int dk' G(k') H(k - k')$$

- **Implications for MRI:**
  - **Filtering effects can be understood either by multiplying the filter in image domain or convoluting with the FT form of the filter in** $k-space$
  - **Important for understanding finite sampling (sampling + windowing) effects in DFT**

Continuous FT Properties

- **Shift Theorem:**

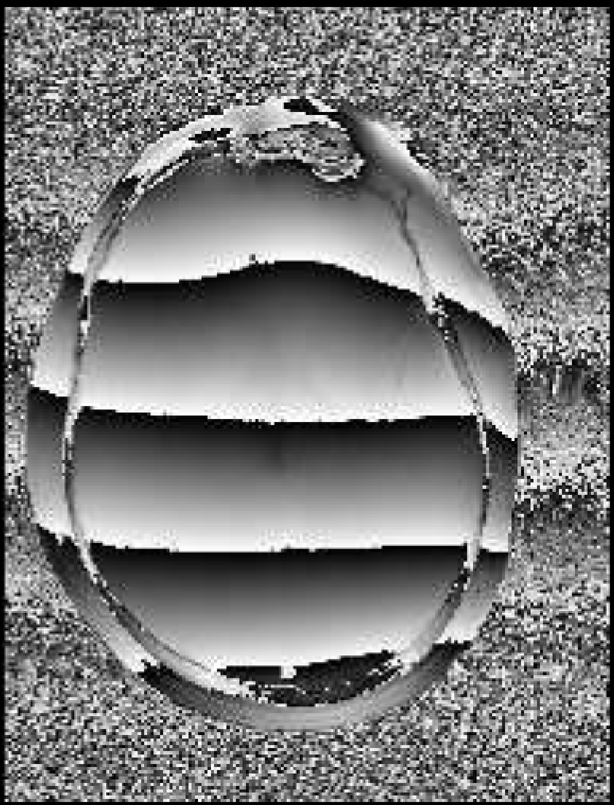
Shift in one domain <=> Additional Linear phase in another

e.g. $s_m(k) \rightarrow s_m(k - k_0) \Rightarrow \hat{\rho}(x) \rightarrow \hat{\rho}(x)e^{i2\pi k_0 x}$

$s_m(k) \rightarrow s_m(k)\, e^{-i2\pi k x_0} \Leftarrow \hat{\rho}(x) \rightarrow \hat{\rho}(x - x_0)$

- **Implications for MRI:**
  - **Motion/displacement of the object will introduce phase error between k-space lines, introducing artifacts (Chap.23)**
  - **A rigid body motion induced phase error can be recovered by adjusting the phase of corresponding k-space lines**
  - **Over coverage in the k-space center is necessary to ensure a high signal baseline in the image**

Shift in k-space



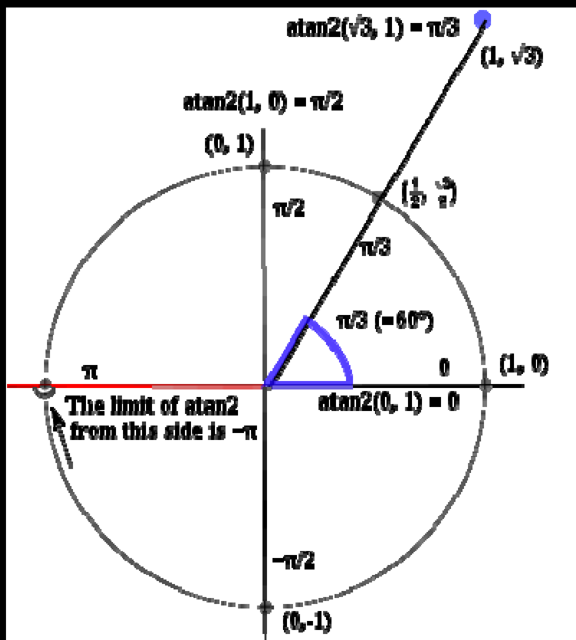
One point

$\hat{\rho}(x)$ -> $\hat{\rho}(x)e^{i2\pi k_0 x}$

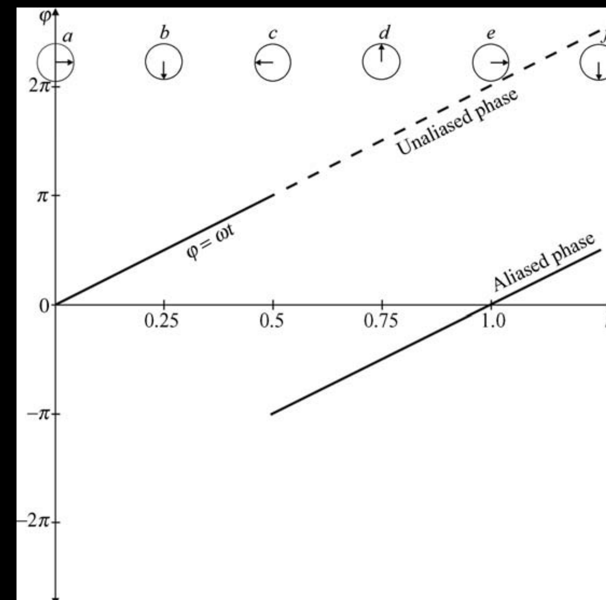
$\hat{\rho}(n\Delta x)$ -> $\hat{\rho}(n\Delta x)e^{i2\pi \cdot m\Delta k_0 \cdot n\Delta x}$

How many points?

Phase Aliasing



$$\phi = \tan2^{-1}\frac{Imaginary}{real}$$



arctan2 function

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\frac{y}{x} & x > 0 \\ \arctan\frac{y}{x} + \pi & y \geq 0, x < 0 \\ \arctan\frac{y}{x} - \pi & y < 0, x < 0 \\ +\frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

Takes into consideration the sign of the individual real and imaginary parts to place the vector in the right quandrant

Images source: Wikipedia.org

Signal sampling function

$$u(k) = \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) = \Delta k \sum_{p=-\infty}^{\infty} \mathcal{F}[e^{i2\pi p\Delta kx}]$$

$$\Downarrow$$

$$U(x) = \mathcal{F}^{-1}\,[u(k)] = \Delta k \sum_{p=-\infty}^{\infty} e^{i2\pi p\Delta kx}$$

$$\Downarrow$$

$$\textstyle\sum_{n=-\infty}^{\infty} e^{i2\pi na} = \sum_{m=-\infty}^{\infty} \delta(a-m)$$

$$\delta(ax) = \delta(x)/|a|$$

$$U(x) = \sum_{q=-\infty}^{\infty} \delta(x - \frac{q}{\Delta k})$$

Infinite sampling

- **Sampling function:**

$$u(k) = \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k)$$

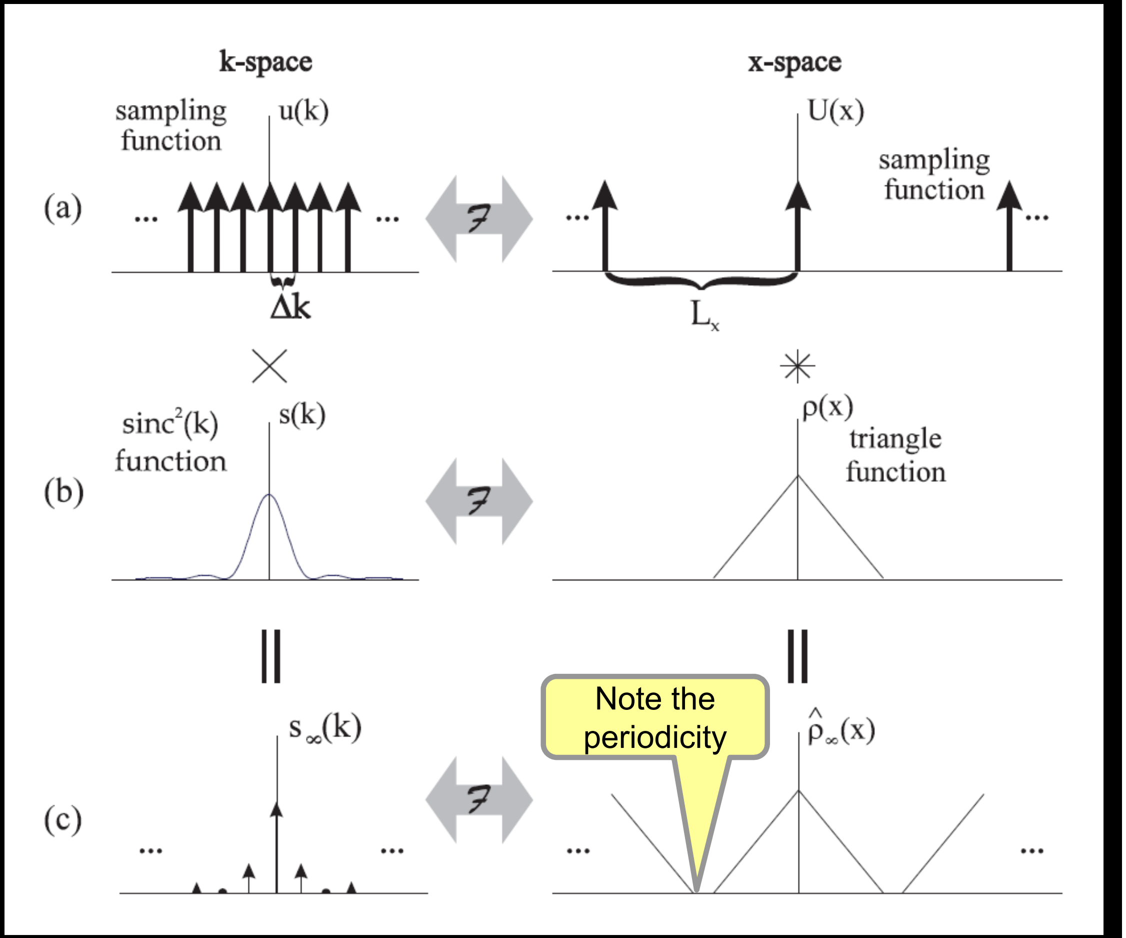
$$U(x) = \sum_{q=-\infty}^{\infty} \delta(x - q/\Delta k)$$

- **Sampled signal:**

$$s_\infty(k) \equiv s(k)u(k)$$
$$= \Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k)\delta(k - p\Delta k)$$

- **Spin density after DFT**

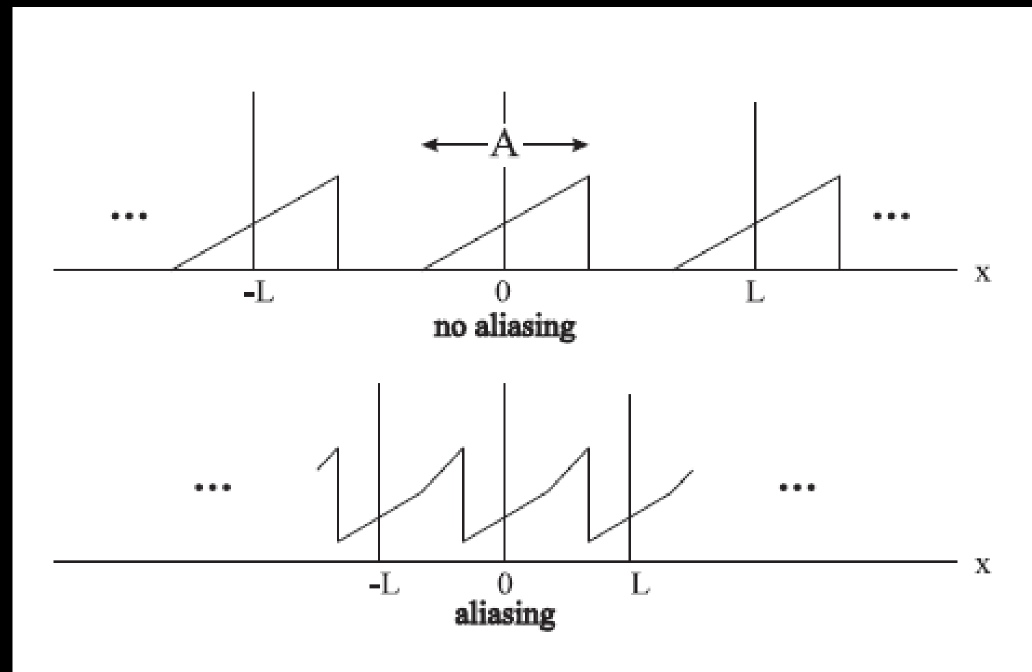
$$\hat{\rho}_\infty(x) = \rho(x) * U(x)$$
$$= \sum_{q=-\infty}^{\infty} \rho(x - q/\Delta k)$$

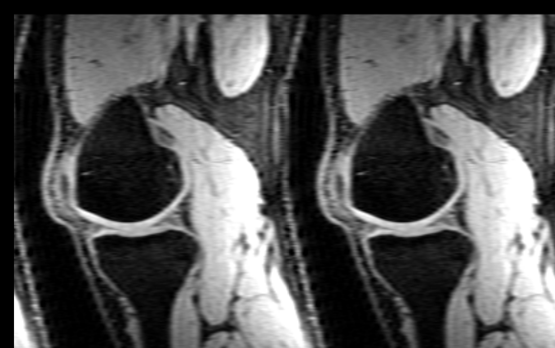


Aliasing and Nyquist Sampling Criterion

- **$1/\Delta k = L$**
- **If object length (A) >L, then overlaps in $\hat{\rho}_{\infty}(x)$ will take place, introducing aliasing artifacts**
- **To avoid aliasing:**
  A<L or $\Delta k<1/A$

Nyquist Sampling Criterion for MRI





Nyquist in Read & Phase encoding directions

$$\left.\begin{array}{c}\Delta k_R = \gamma G_R \Delta t \\ \Delta k_R = 1/L_R \\ \Delta t = T_s/N = 1/BW_R\end{array}\right\} \quad BW_R = \gamma G_R L_R > \gamma G_R A_R$$

$$\left.\begin{array}{c}\Delta k_{PE} = \gamma \Delta G_{PE} \tau_{PE} \\ \Delta k_{PE} = 1/L_{PE} \\ \Delta t = TR = 1/BW_{PE}\end{array}\right\} \quad \gamma \Delta G_{PE} \tau_{PE} = \frac{1}{L_{PE}} < \frac{1}{A_{PE}}$$

Finite, discrete sampling

- **Data truncation**

$$s_{m}(k) = s(k) \cdot u(k) \cdot rect(\frac{k+\Delta k/2}{W})$$

$$\hat{\rho}(x) = \rho(x) * U(x) * W sinc(\pi W x) e^{-i\pi x \Delta k}$$

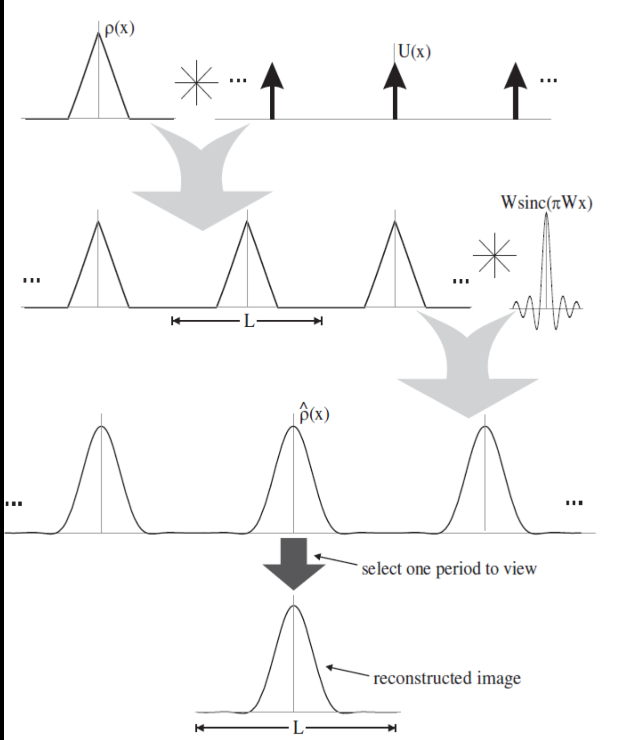
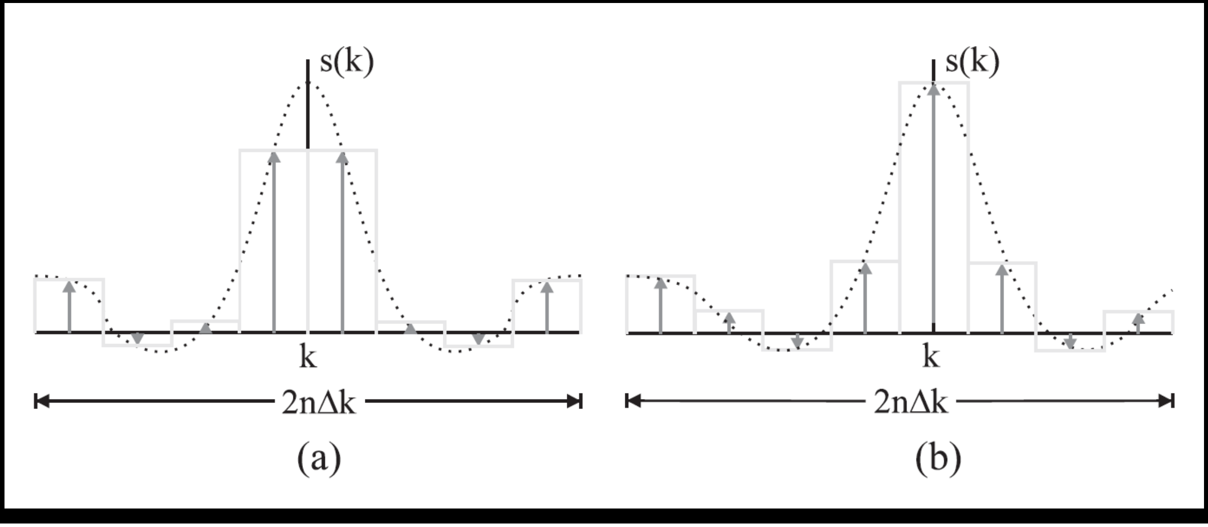


Fig. 12.6



Discrete Fourier Transform(DFT)

$$s(p\Delta k) = \Delta x \sum_{q=-n}^{n-1} \hat{p}(q\Delta x) e^{-i\pi pq/n}$$

$$\hat{p}(q\Delta x) = \Delta k \sum_{q=-n}^{n-1} s(p\Delta k) e^{i\pi pq/n}$$

- **Resolution (in voxel size)**

$$\Delta L_{RO} = \frac{BW_R}{N_R \gamma G_R}$$

$$\Delta L_{PE} = \frac{1}{N_{PE} \gamma \Delta G_{PE} \tau_{PE}}$$

This class - Key points

- Filtering effects
- Effective voxel size
- Spin echo and gradient echo envelope
  – filtering affects

FT image reconstruction

- **Spatial encoded signal and spin density**

Continuous:

$$s(k_x, ky, kz) = \iiint dxdydz\ \rho(x, y, z)e^{-i2\pi(k_x x + k_y y + k_z z)}$$

$$\rho(x, y, z) = \iiint dk_x dk_y dk_z\ s(k_x, k_y, k_z)e^{i2\pi(k_x x + k_y y + k_z z)}$$

Infinite discrete sampling:

$$\hat{\rho}_\infty(x) = \Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k)e^{i2\pi p\Delta k x}$$

Finite discrete sampling:

$$\hat{\rho}_w(x) = \Delta k \sum_{p=-n}^{n-1} s(p\Delta k)e^{i2\pi p\Delta k x}$$

$$= \hat{\rho}_\infty(x) * W sinc(\pi W x)e^{-i\pi x \Delta k}$$ **(*W* = *NΔk*)**

Ideal voxel size:

$$\Delta x = \frac{L}{N} = \frac{1}{N\Delta k} = \frac{1}{W}$$

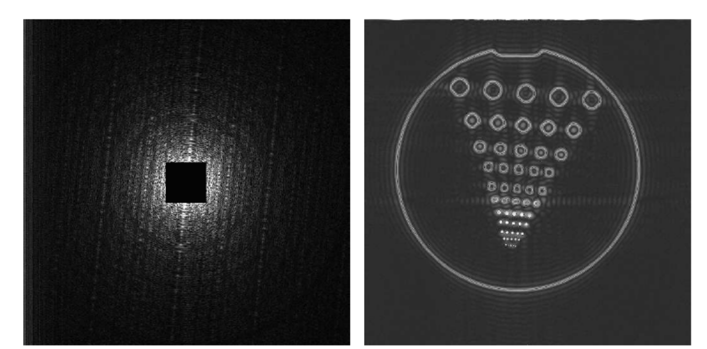
Filters and Point Spread Function (PSF)

- Filter: H(k) multiplies with k-space
- PSF: iFT of H(k), i.e. h(r), convolve with the image

- According to Convolution Theorem:

$s_H(k) = s(k) \cdot H_1(k) \cdot H_2(k)$ ….. --> $\rho_H(x)=\rho(x)*h_1(x)*h_2(x)$…

- Examples of filter:
  - K-space truncation (HP filter)
  - Discrete sampling (mild LP filter)
  - T2/T2' relaxation (LP filter)



(a) (b)

Gibbs Ringing: examples

- **Note the coherent addition of the ringing**

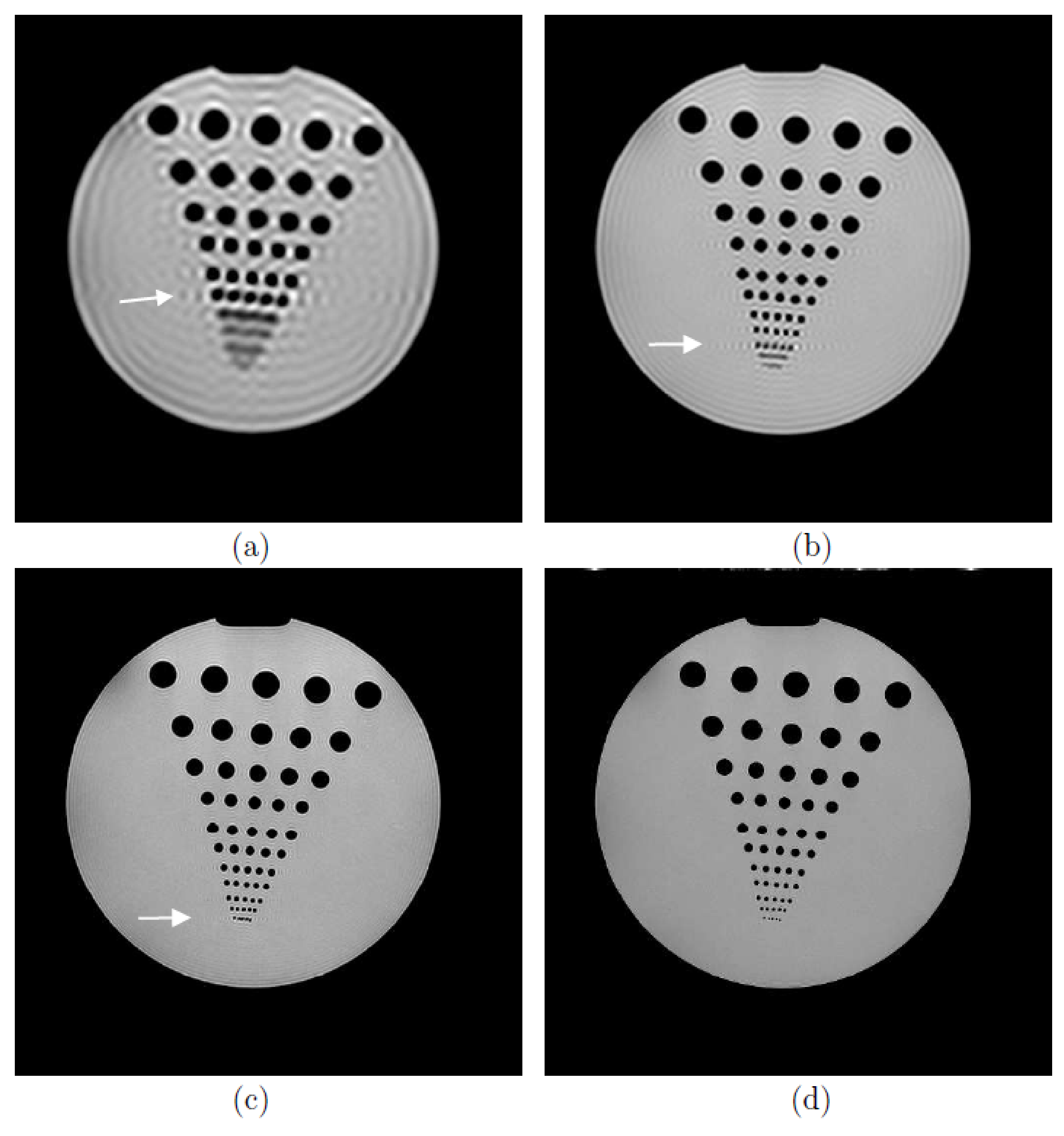
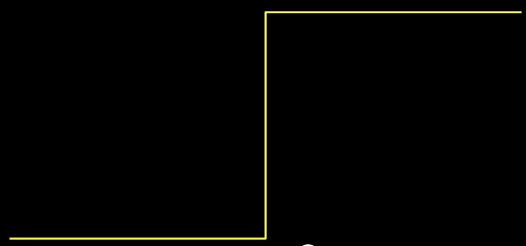


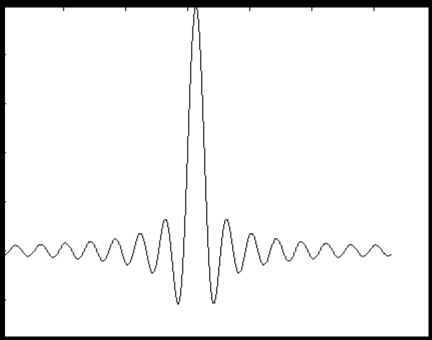
Fig.15.5

Gibbs Ringing



$x_0 = 0$

*



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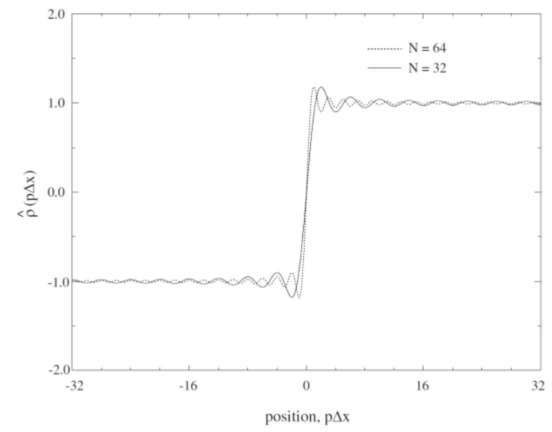


- **Discontinuity in the object**

$$f(x) = fc(x) + (f(0^+) - f(0^-))\Theta(x)$$

- **Symmetric sampling window on k-space**

$$H_w^{sym}(k) = rect(k/W) \textbf{ => } h_w^{sym}(x) = W sinc(\pi W x)$$

- **Reconstructed image**

$$\hat{f}(x) = f(x) * h_w^{sym}(x)$$

Gibbs Ringing: oscillation amplitude limit

$$\hat{f}(x) = f(x) * h_w^{sym}(x)$$

- **For fixed FOV and $\mathrm{N} \to \infty$, one has:**

$$W = \frac{1}{\Delta x} \to \infty$$

- **The limit of $\hat{f}(x)$ is:**

$$\lim_{W\to\infty} \hat{f}(x) = f_c(x) \lim_{W\to\infty} \int_{-\infty}^{\infty} h_w^{sym}(x-x')dx' + \left(f(0^+) - f(0^-)\right) \lim_{W\to\infty} \int_{x'=0}^{\infty} h_w^{sym}(x-x')dx'$$

$$= f_c(x) + \left(f(0^+) - f(0^-)\right) \lim_{W\to\infty} \int_{x_0=0}^{\infty} h_w^{sym}(x-x')dx'$$

$$= fc(x) + \left(f(0^+) - f(0^-)\right) \lim_{W\to\infty} \left(\frac{1}{2} + \frac{1}{\pi} Si(\pi W x)\right)$$

- **The largest oscillation takes place at Δx, so that**

$$Si(\pi W \Delta x) = Si(\pi) \cong 1.8519$$

thus

$$\lim_{W\to\infty} \hat{f}(\Delta x) \cong f(\Delta x) + 0.09(f(0^+) - f(0^-))$$

Gibbs Ringing: oscillation frequency

- $\hat{f}(x) = f(x) * h_w^{sym}(x)$
$$= f(x) * W sinc(\pi W x)$$
$$= f(x) * W sinc\left(\frac{\pi x}{\Delta x}\right)$$
$$= f(x) * W sinc(q\pi)$$

- **Spatial frequency of the Gibbs Ringing depends only on pixel size Δx**

- **Higher resolution -> smaller pixels -> denser Gibbs Ringing but slightly greater signal variation**

Gibbs Ringing

- **Takes place at step discontinuities**

- **Oscillating over- and under-shoots, with constant amplitude limit**

$$\lim_{N\to\infty}\left|\hat{\rho}\left(x_0^{\pm}\right)-\rho\,(x_0^{\pm})\right|\approx 0.09|\rho\,(x_0^{+})-\rho\,(x_0^{-})|$$

- **Oscillating frequency (spatial) depends only on pixel number away from the discontinuity**

Gibbs Ringing: reduction

- Properties of Gibbs Ringing
  - Proportional to the signal difference at the discontinuity
  - Ringing variation in amplitude and frequency is a function of pixel number
  - Over- and under-shoot alternates by every other pixel
  - The less sharper the discontinuity, the less obvious the Gibbs Ringing

- Gibbs Ringing reduction
  - LP filter (e.g. Hanning, Hamming, Gaussian) to smooth out the image

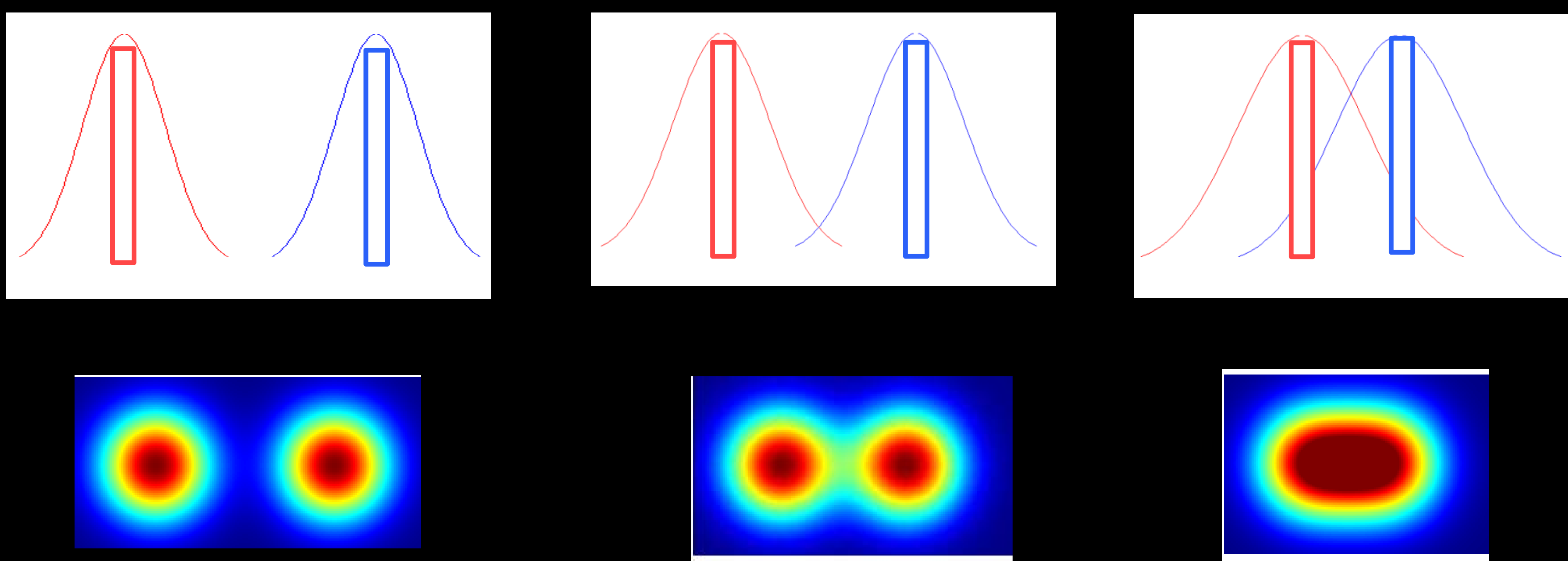
Spatial Resolution in MRI

- **Definition**

The smallest resolvable distance between two different objects/features

- **PSF (if known) can be used to quantify resolution limit**

Ideal case: PSF = delta function



PSF and spatial resolution

- **Spatial resolution of PSF (continuous filter)**

$$\Delta x_{filter} \equiv \frac{1}{h_{filter}(0)} \int_{-\infty}^{\infty} dx h_{filter}(x) = \frac{H_{filter}(0)}{h_{filter}(0)}$$

- **Spatial resolution of discrete, windowed, sampled MR signal**

$$\Delta x_{MRI,filter} \equiv \frac{1}{h_{ws,filter}(0)} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, h_{ws,filter}(x)$$

$\Downarrow$ w/o additional filter

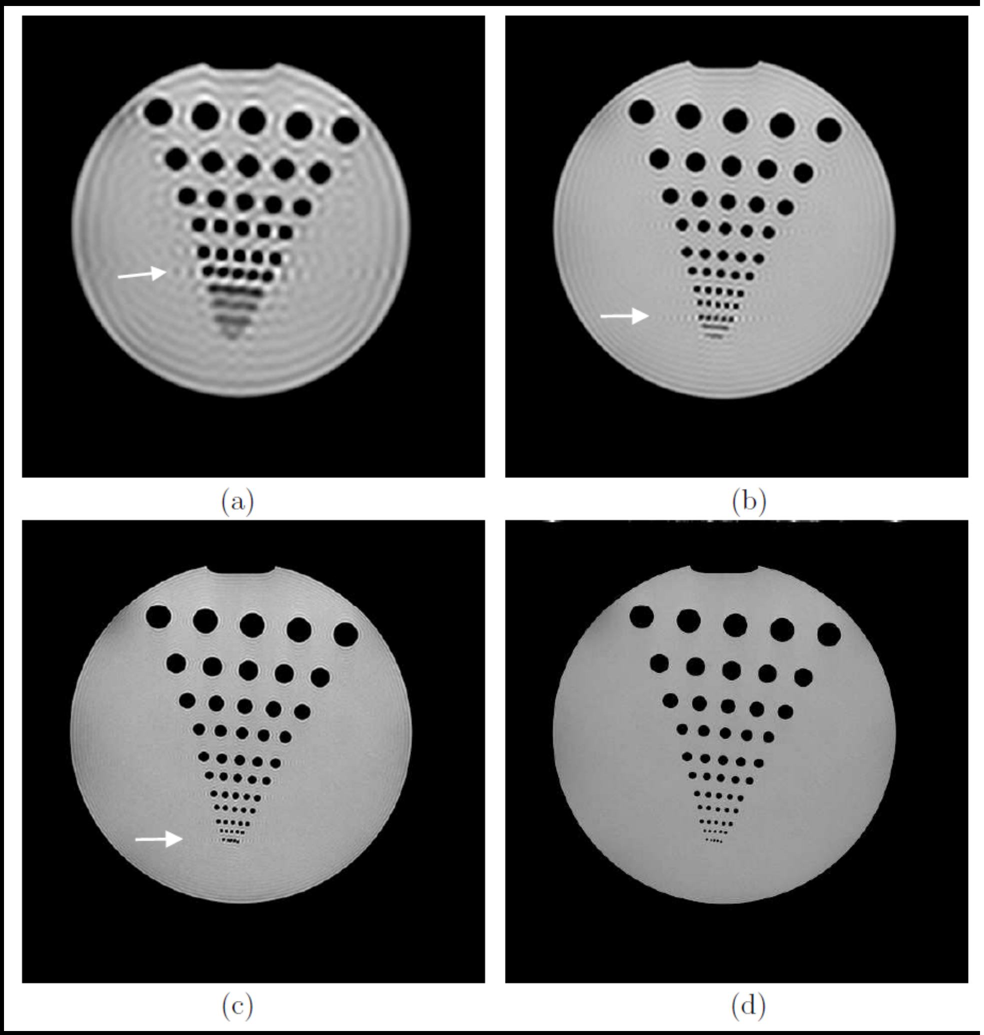
$$\Delta x_{MRI,ws} = \frac{1}{h_{ws}(0)} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, h_{ws}(x) = \frac{1}{2n\Delta k} = \Delta x_{FT}$$

- **Conditions:**
  - **H_{filter} is symmetric**
  - **$H_{filter}(0) \geq H_{filter}(k \neq 0)$**

K-space coverage and spatial resolution

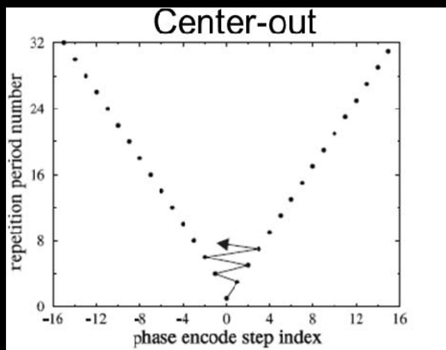
$$\frac{1}{2n\Delta k} = \frac{1}{W} = \Delta x = \frac{L}{2n}$$

- (fixed L)
  2nΔk
  -> Δx ↓
  -> Gibbs Ringing↓
  -> SNR ↑



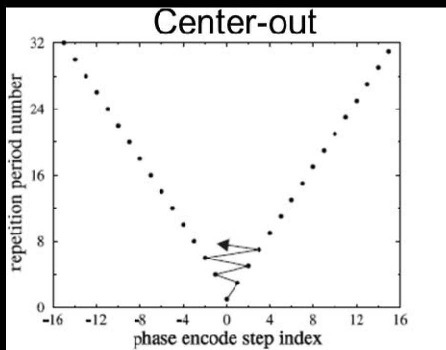
Quick Recap:

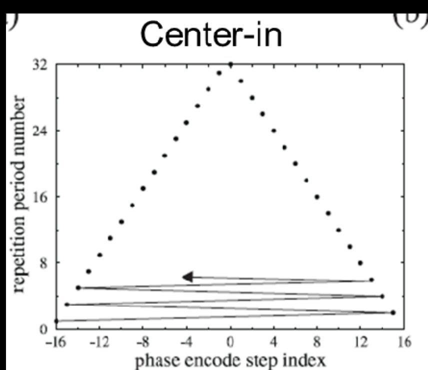
Phase encoding order

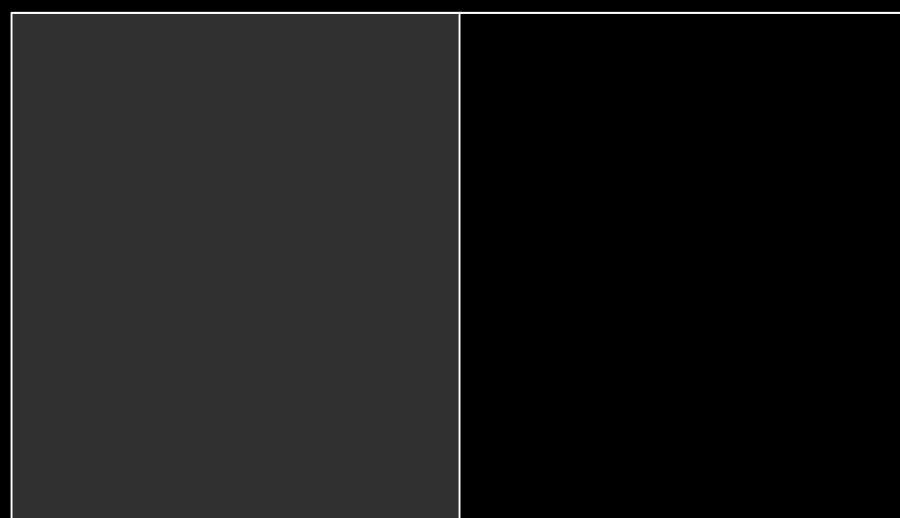


Quick Recap:

Phase encoding order

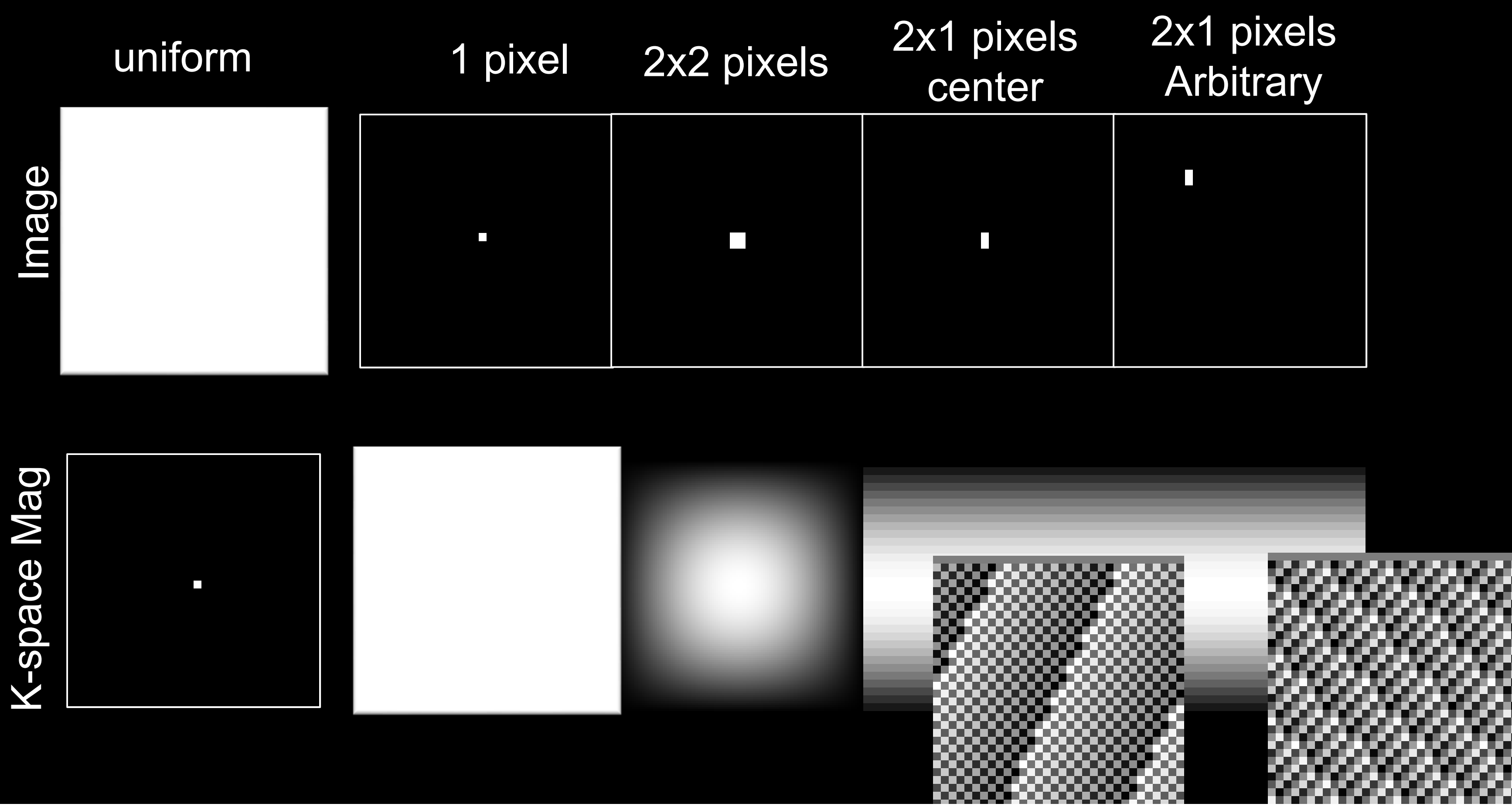




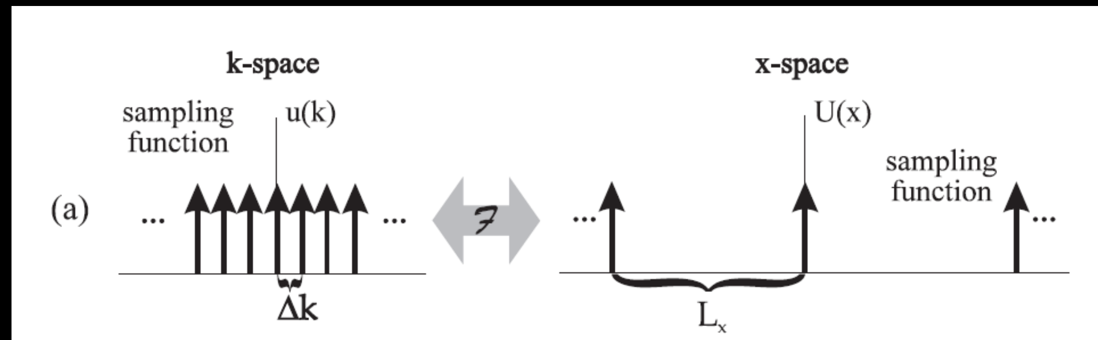


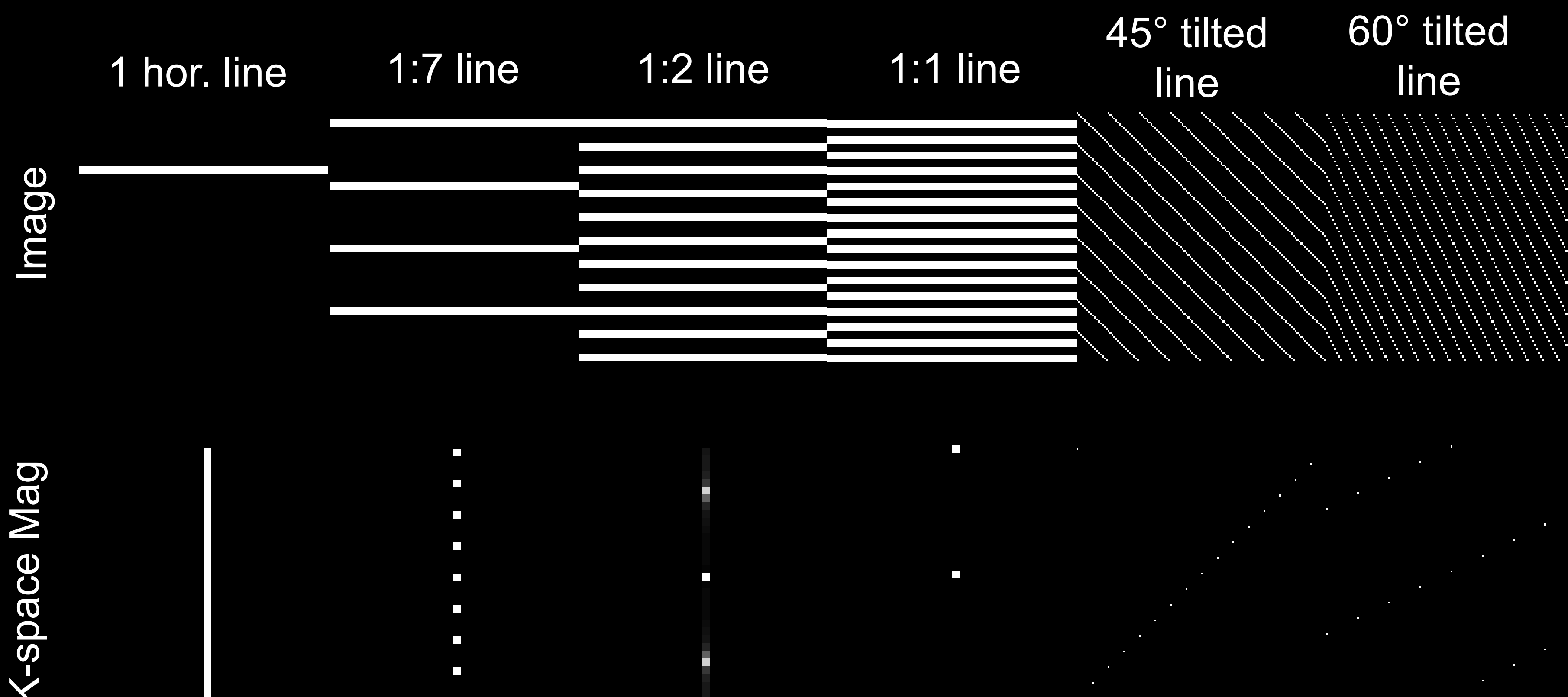
Information content in k-space

- K-space center: low spatial frequency components
- K-space edge : high spatial frequency components



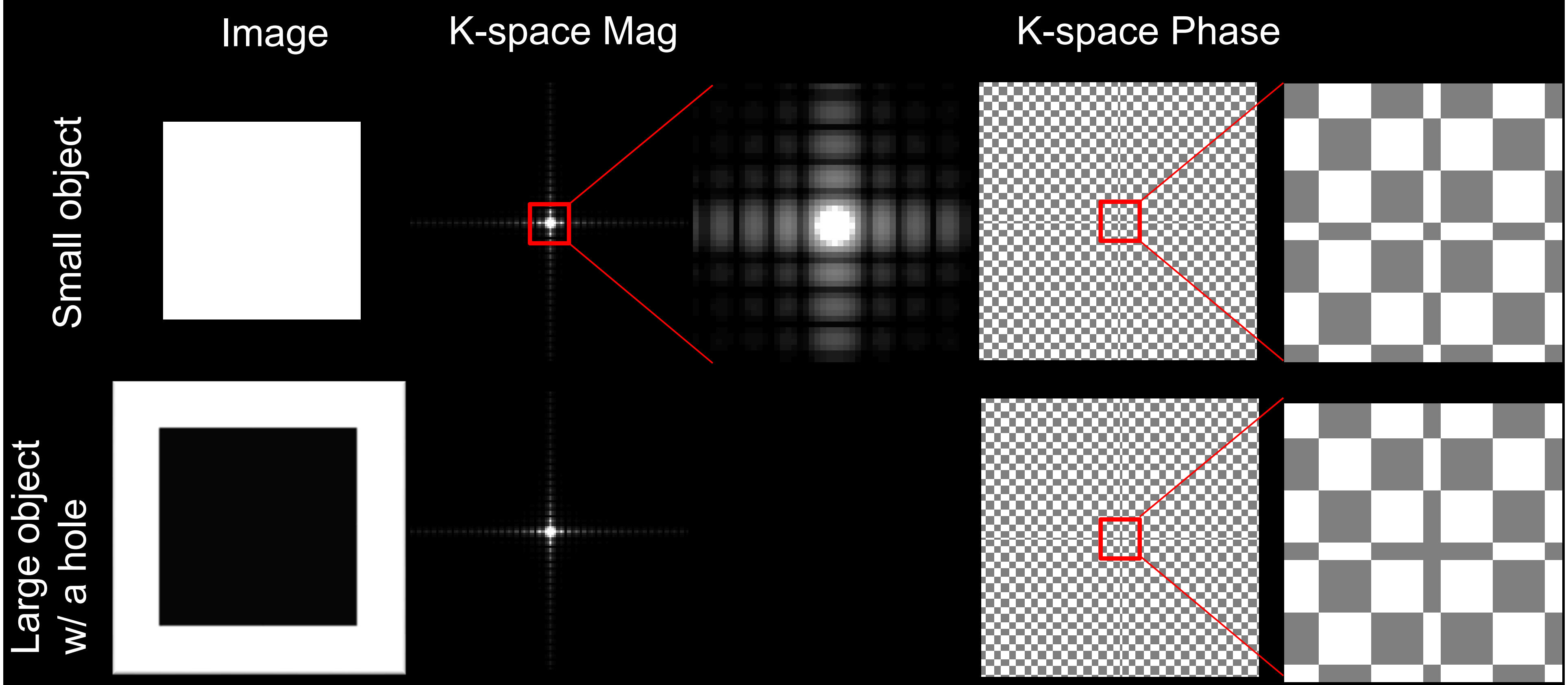
Information content in k-space





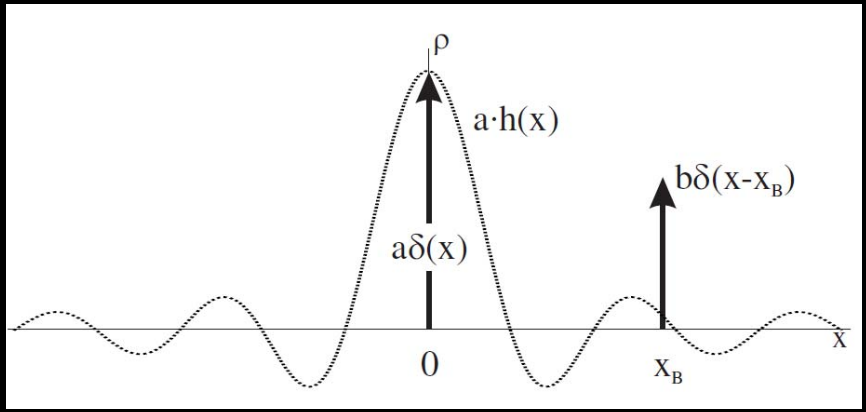
Information content in k-space

- **K-space phase**

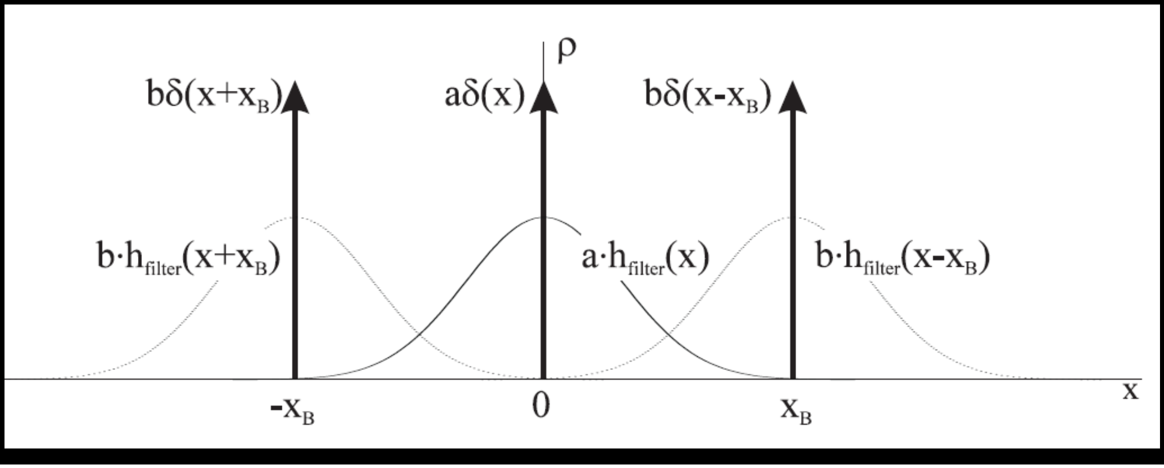


Other measures of resolution

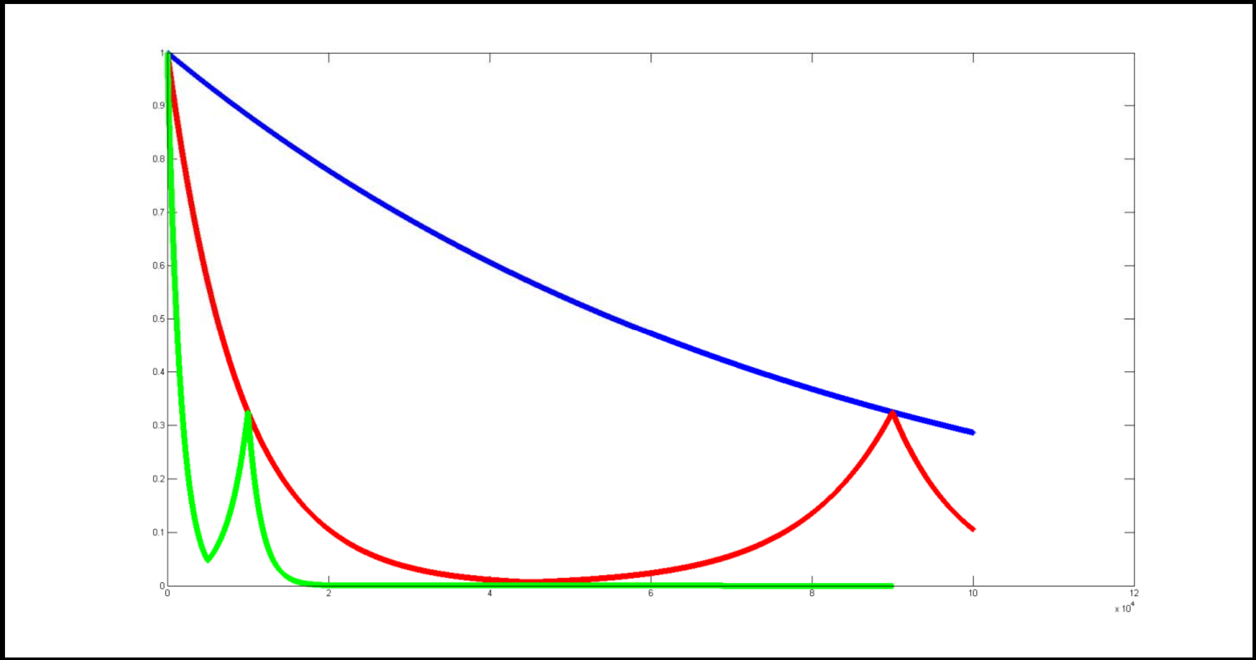
- First zero crossing of the filter h(x), e.g. Sinc



- Full Width Half/Tenth Maximum approximation



Filtering due to T2 and T2* decay



- Posing an intrinsic resolution limit for MR images
- Depend on sequence and scanning parameters in a complicated way (Chap. 15)

Filtering due to T2 and T2* decay

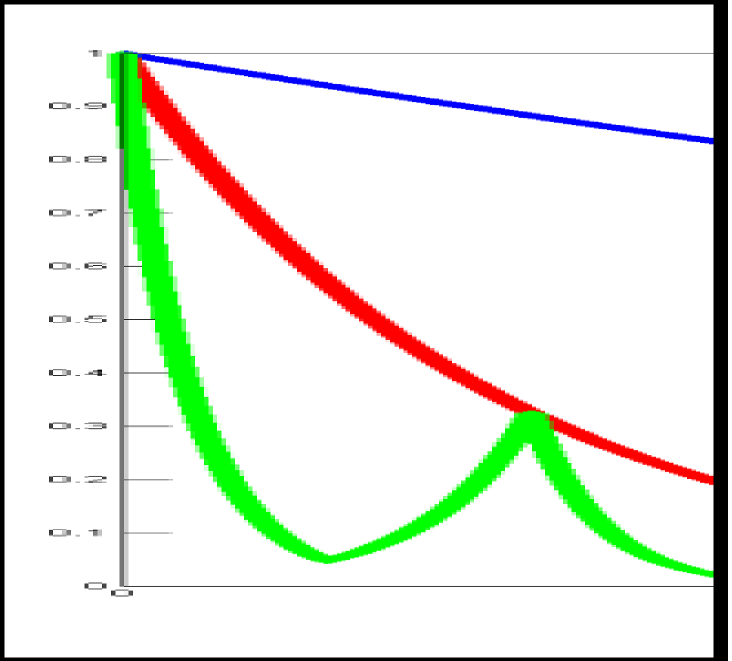
- **Along Read direction**

$$H_{w(sym),T_2^*}(k) = e^{-\frac{\frac{k}{\gamma G}+TE}{T_2^*}} Rect(\frac{k}{W})$$

iFT is Lorentzian function

iFT is Sinc function

$$h_{w(sym),T_2^*}(x) = Eq\ (13.54)$$

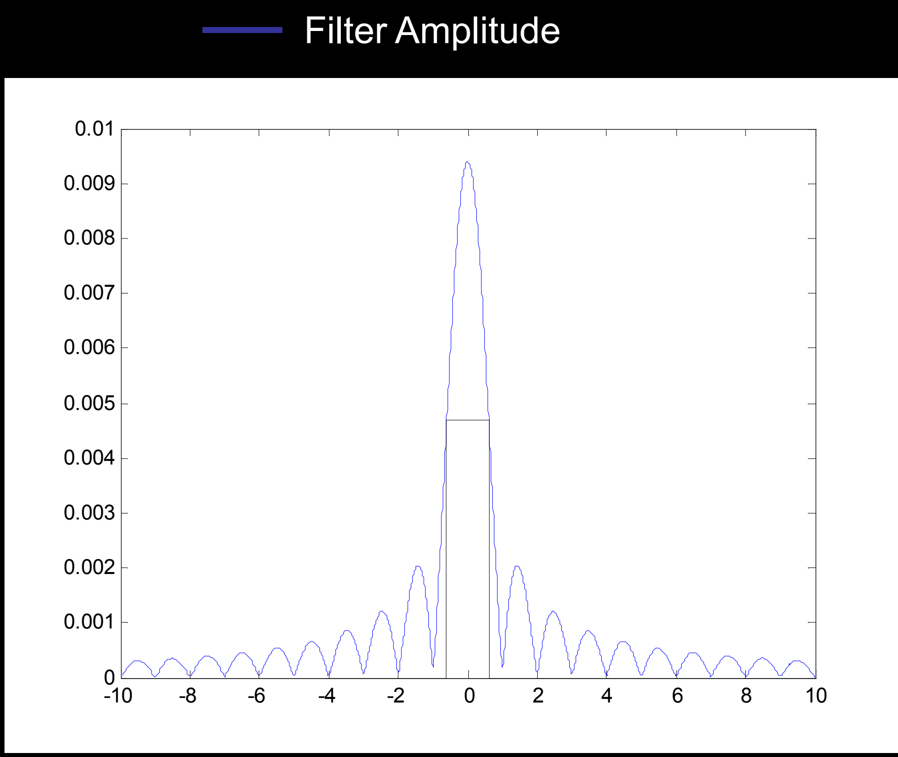


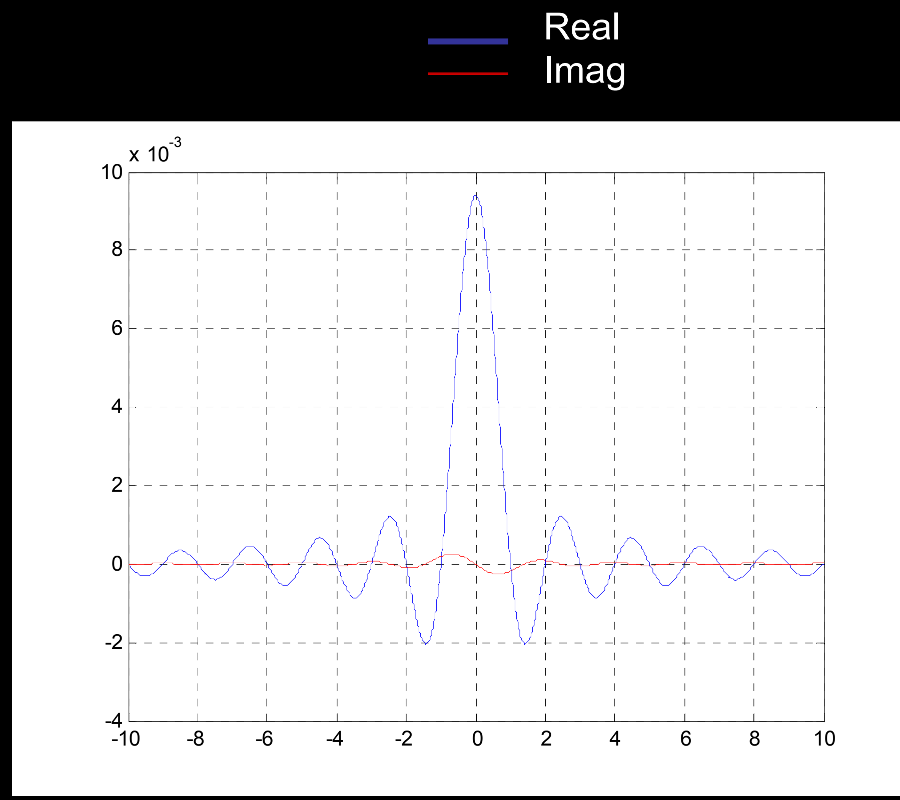
- **When in effect?**
  - **Long RO relative to T2 or T2***
- **Pros and Cons**
  - **Limit the effective resolution, even blur the image**
  - **Reduce Gibbs ringing**

T2* filtering affect

$$h_{w,T_2^*}(x) = e^{-\frac{TE}{T_2^*}} \cdot \frac{e^{\frac{T_s}{2T_2^*}-\frac{i\pi x}{\Delta x}} - e^{\frac{-T_s}{2T_2^*}+\frac{i\pi x}{\Delta x}}}{\frac{T_s \Delta x}{T_2^*} - i2\pi x}$$

$$\frac{T_s}{T_2^*} = 1/8$$





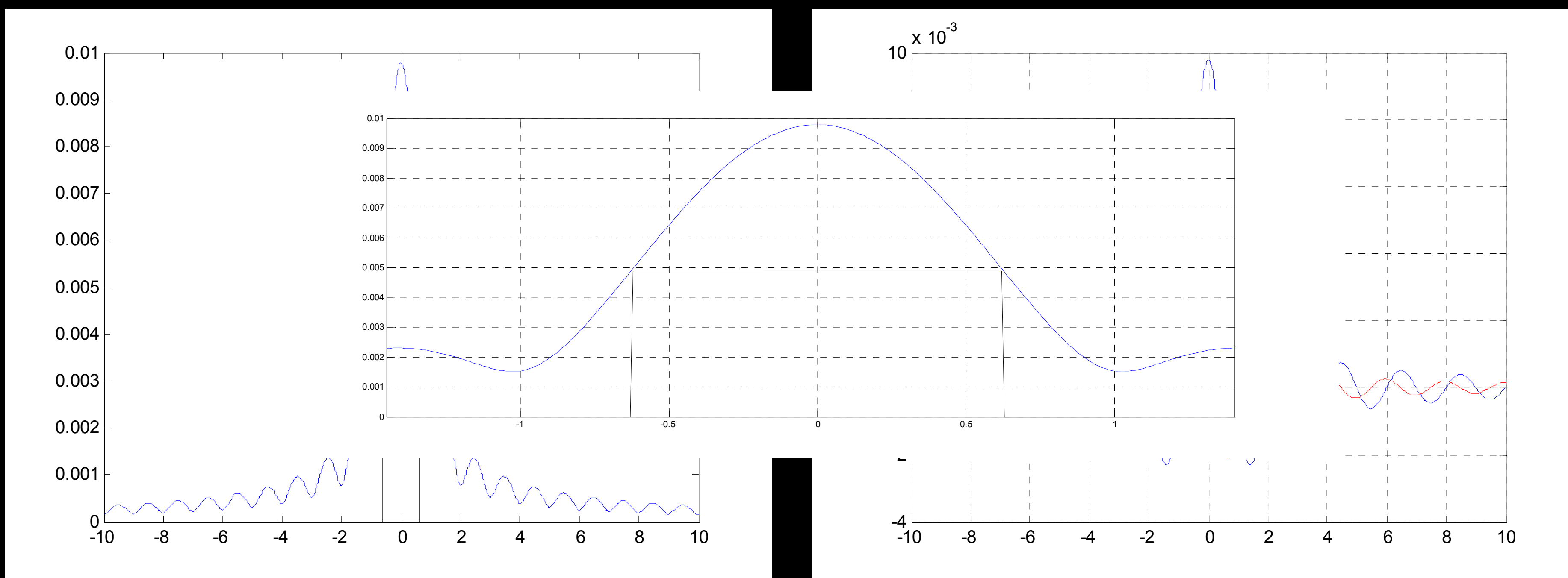
T2* filtering affect

$$h_{w,T_2^*}(x) = e^{-\frac{TE}{T_2^*}} \cdot \frac{e^{\frac{T_s}{2T_2^*}-\frac{i\pi x}{\Delta x}} - e^{\frac{-T_s}{2T_2^*}+\frac{i\pi x}{\Delta x}}}{\frac{T_s \Delta x}{T_2^*} - i2\pi x}$$

$$\frac{T_s}{T_2^*} = 1$$

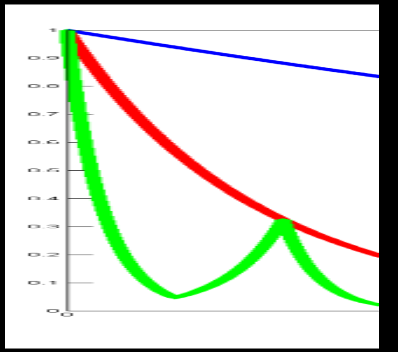
Filter Amplitude

Real
Imag



T2' filtering affect

$$
\begin{aligned}
H_{SE}(k) &= H_{SE,T_2}(k) H_{SE,T_2'}(k) \\
&= e^{-T_E/T_2} e^{-k/\gamma G T_2} e^{-|k|/\gamma G T_2'}
\end{aligned}
$$



$$H_{w,SE,T_2'}(k) = e^{-|k|/(\gamma G T_2')} \mathrm{rect}\left(\frac{k + \Delta k/2}{W}\right)$$

$$h_{SE,T_2'}(x) = \frac{2\gamma G T_2'}{1 + 4\pi^2 \gamma^2 G^2 T_2'^2 x^2}$$

$$\mathrm{FWHM}_{SE} = \mathrm{FWHM}_{SE,T_2} + \mathrm{FWHM}_{SE,T_2'}$$

Homework

- *Prob 13.1,13.5, 13.7, 13.8*

Next Session

Review