Ch. 4

- Magnetization **M**
- Bloch equations of motion: $d\mathbf{M}/dt = \gamma \mathbf{M} \times \mathbf{B}$
- Potential energy of magnetic moment:

  $E = -\mu \cdot \mathbf{B}$
- T_1 recovery:

  $M_z(t_0)\, e^{-(t-t_0)/T_1} + M_0\,(1 - e^{-(t-t_0)/T_1})$
- Longitudinal (spin lattice) relaxation time T_1
- Transverse (spin-spin) relaxation time T_2
- Dephasing

Magnetization

- Magnetization **M** is the sum of magnetic moments per unit volume: $\mathbf{M} = (\Sigma\, \mu)/V$
- In most chapters of the MRI textbook, the magnetization is from protons.
- $d\mu/dt = \gamma\mu \times \mathbf{B}$ becomes $d\mathbf{M}/dt = \gamma\mathbf{M} \times \mathbf{B}$
- When $\mathbf{B} = B_0\, \mathbf{z}$, the Bloch equation becomes: $dM_z/dt = 0$ and $d\mathbf{M}_a/dt = \gamma\mathbf{M}_a \times \mathbf{B}$

Potential energy

- Potential energy of one spin: $E = -\mu \cdot \mathbf{B}$
- This equation implies that a magnetic moment will tend to align itself to the magnetic field in order to achieve minimum energy.
- Potential energy of spins per given volume: Energy density $U = -\mathbf{M} \cdot \mathbf{B}$
- Curie's law:

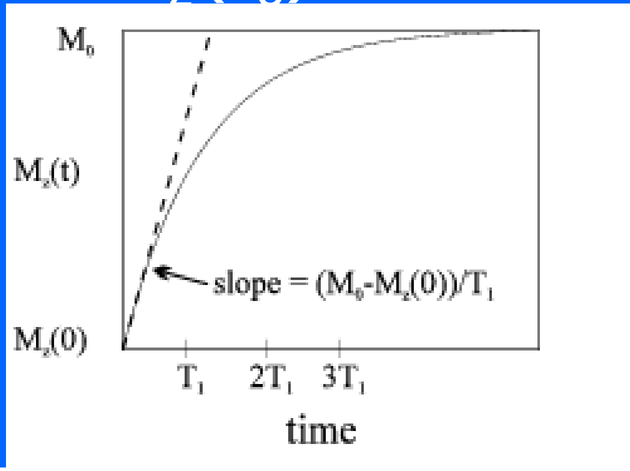
  equilibrium magnetization $\mathbf{M}_0 = C\,(\mathbf{B}/T)$ or $\mathbf{M}_0$ ó $1/T$, where T is the temperature.

T_1 relaxation

- A magnetic moment tends to be parallel to a magnetic field (in order to minimize the energy). Such an interaction is the spin lattice interaction. The (recovery or growth) rate is $1/T_1$, where T_1 is the experimental relaxation time:

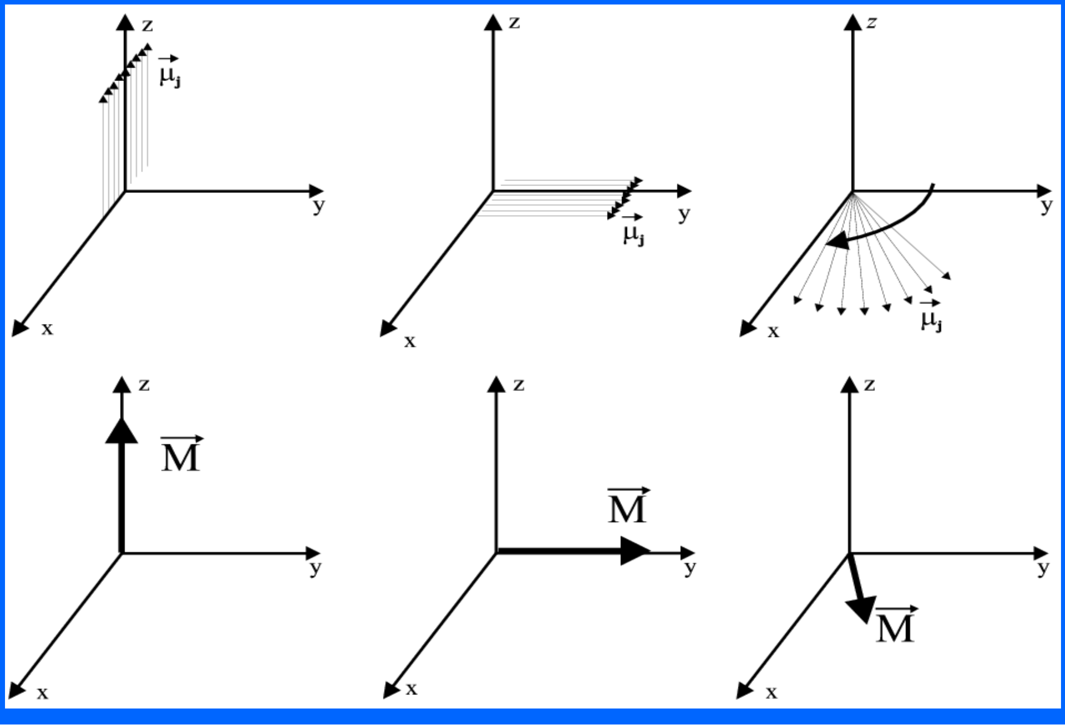
  $dM_z/dt = (M_0 - M_z)/T_1$ (when **B** // **z**)

- Solution: $M_z(t) = M_z(t_0)\, e^{-(t-t_0)/T_1} + M_0\,(1 - e^{-(t-t_0)/T_1})$



Spin-spin interaction

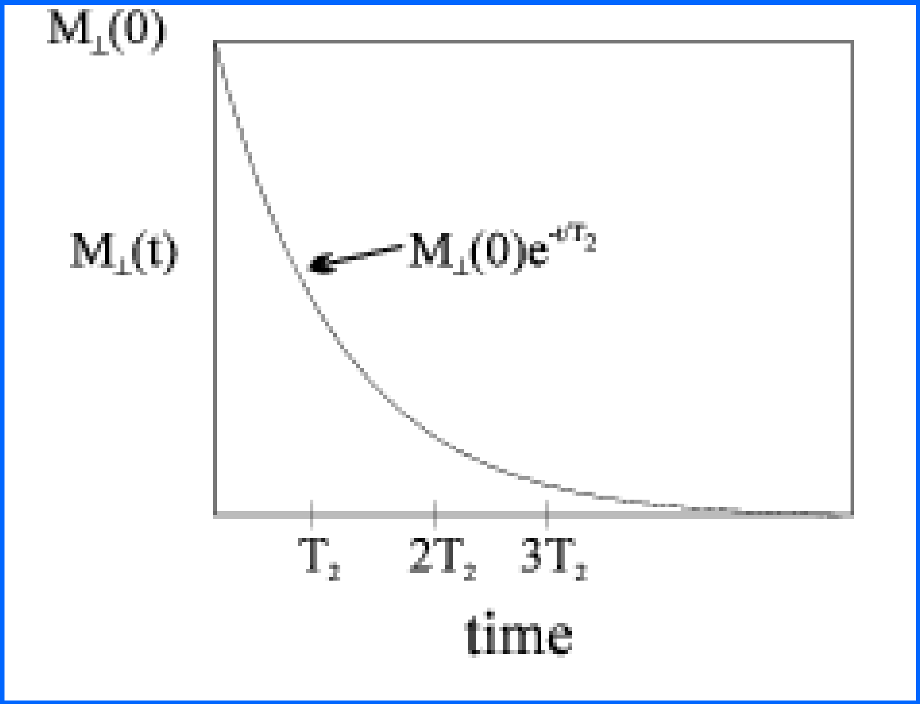
- Due to the magnetic field produced by each spin, the spin-spin interaction is related to local, random, and time-dependent field variations. Such a microscopic interaction leads to a macroscopic T_2 decay.



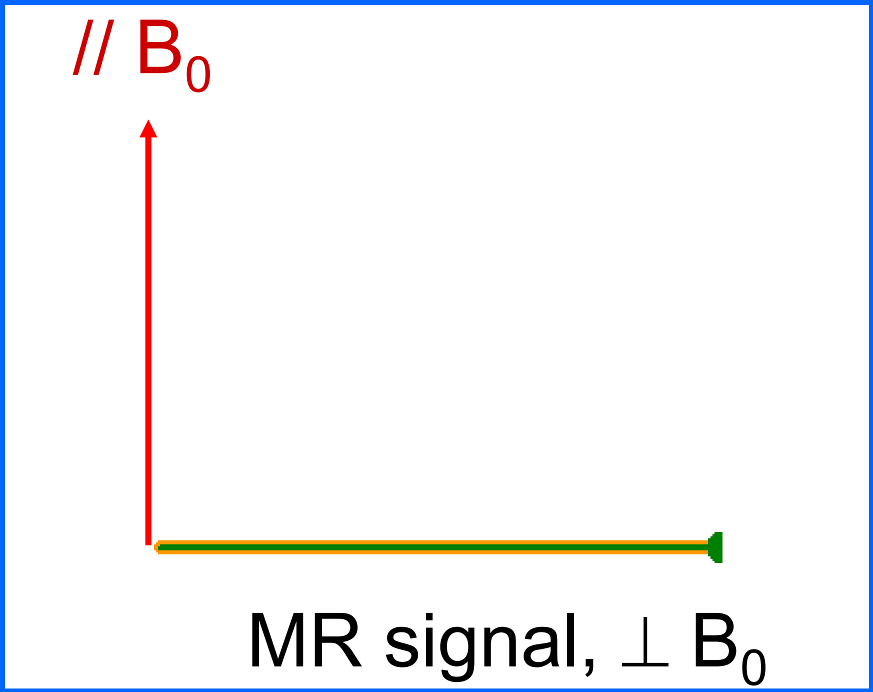
T_2 decay

- $d\mathbf{M}_a/dt = \gamma \mathbf{M}_a \times \mathbf{B} - \mathbf{M}_a/T_2$
- $(d\mathbf{M}_a/dt)' = -\mathbf{M}_a/T_2 \Rightarrow \mathbf{M}_a(t) = \mathbf{M}_a(t_0)\, e^{-(t-t_0)/T_2}$

T_2 decay



T_2 decay and T_1 regrow



T_1 and T_2 of some tissues

Tissue	T_1 (ms)	T_2 (ms)
gray matter (GM)	950	100
white matter (WM)	600	80
muscle	900	50
cerebrospinal fluid (CSF)	4500	2200
fat	250	60
blood	1200	100-200

T_2' and T_2^*

- External magnetic fields can cause extra dephasing of spins. This leads to T_2' decay.
- The T_2' decay can be recovered by the spin echo sequence (more in Ch. 8).
- The "total T_2" is now called T_2^*, which can be calculated from $1/T_2^* = 1/T_2' + 1/T_2$
- The reason we add the inverse of relaxation times is due to the form of the Bloch equation.
- Note that "relaxation rates" are defined as: $R_2 = 1/T_2$, $R_2' = 1/T_2'$, and $R_2^* = 1/T_2^*$

Static field solutions

- The complete Bloch equation is:

$$d\mathbf{M}/dt = \gamma\mathbf{M} \times \mathbf{B}_{eff} + (M_0 - M_z)/T_1\,\mathbf{z} - \mathbf{M}_a/T_2$$

- Consider $\mathbf{B} = B_0\,\mathbf{z}$
- $dM_z/dt = (M_0 - M_z)/T_1$
- $dM_x/dt = \omega_0 M_y - M_x/T_2$
- $dM_y/dt = -\omega_0 M_x - M_y/T_2$
- The solutions are:
- $M_x(t) = e^{-t/T_2}\,(M_x(0)\cos(\omega_0 t) + M_y(0)\sin(\omega_0 t))$
- $M_y(t) = e^{-t/T_2}\,(M_y(0)\cos(\omega_0 t) - M_x(0)\sin(\omega_0 t))$
- $M_z(t) = M_z(0)\,e^{-t/T_1} + M_0\,(1 - e^{-t/T_1})$

Static field solutions: continue

- The x and y components of the solutions can be expressed together by a complex number:
- $M_+(t)$ ¼ $M_x(t) + i\, M_y(t) = e^{-i\omega_0 t - t/T_2}\, M_+(0)$
- or $M_+(t) = |M_+(t)|\, e^{i\phi(t)} = M_a(t)\, e^{i\phi(t)}$

  where $M_a(t) = e^{-t/T_2}\, M_a(0)$

  and $\phi(t) = -\omega_0 t + \phi(0)$
- These notations are commonly used in MRI.

Static and rf fields in the Bloch eq.

- $(dM_z/dt)' = -\omega_1 M_{y'} + (M_0 - M_z)/T_1$
- $(dM_{x'}/dt)' = \Delta\omega\, M_{y'} - M_{x'}/T_2$
- $(dM_{y'}/dt)' = -\Delta\omega\, M_{x'} + \omega_1 M_z - M_{y'}/T_2$
- $\Delta\omega$ ¼ $\omega_0 - \omega$
- For constant $\Delta\omega$, ω_1, T_1, and T_2, the solutions of the Bloch equation can be solved analytically.
- Here in this chapter we are only interested in two special cases:

1. Short-lived rf pulse (with $\omega_1 = 0$ or no relaxation terms)
2. Long-lived rf pulse (derivatives =0)

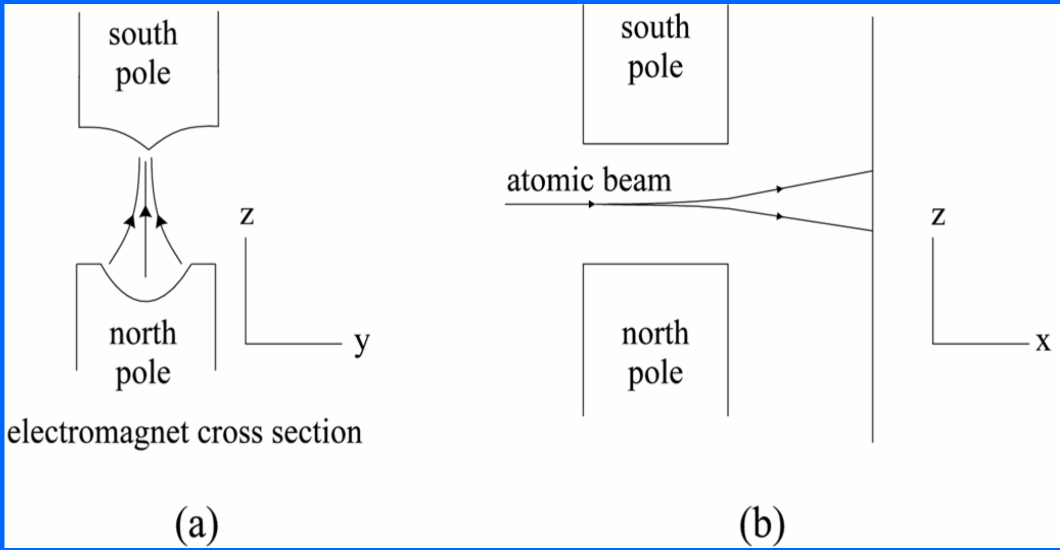
Ch. 5

- Stern and Gerlach experiment
- Zeeman splitting for spin ½ particles
- Transition energy containing the Larmor frequency
- Force on a magnetic moment

Stern and Gerlach experiment

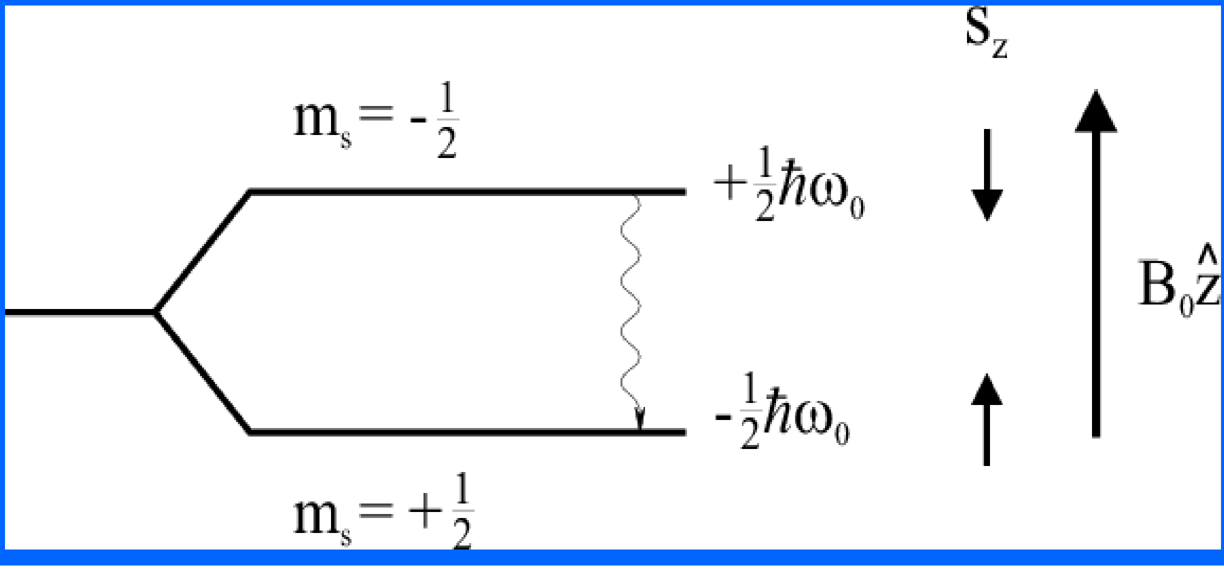
- Neutral silver atom (^{47}Ag) through a magnetic field
- $\mu = \gamma \mathbf{J}$ for electron
- $J_z = m(h/2\pi)$
- $J^2 = j(j+1)\ (h/2\pi)^2$

  $j = 0, 1, \ldots$ or $1/2, 3/2, \ldots$



Zeeman effect

- When an external magnetic field is applied, the atomic or nuclear energy levels are split.
- Spins parallel to the field are at the lower energy level, $-(h/2\pi)\omega_0/2$.
- Spins anti-parallel to the field are at the higher energy level, $+(h/2\pi)\omega_0/2$.
- The frequency ω_0 is the famous Larmor frequency.



Transition energy

- Transition energy is the released energy for a spin jumping from the higher energy level to the lower energy level.
- This transition energy is $(h/2\pi)\omega_0$ where Larmor frequency $\omega_0 = \gamma B$.

Force on a magnetic moment

- Recall potential energy: $E = -\mu \cdot \mathbf{B}$
- Force $\mathbf{F} = -\nabla E = \nabla(\mu \cdot \mathbf{B})$
- Usually μ is not a function of space but magnetization $\mathbf{M}(\mathbf{r})$ is!
- For example, $F_z = \mu_z$ ÖB$_z$/Öz ¼ $\mu_z G_z$
- In MRI, “gradient fields” are usually referred as the derivatives of the B_z component with respect to the spatial direction, i.e., x-gradient is G_x ¼ ÖB$_z$/Öx, y-gradient is G_y ¼ ÖB$_z$/Öy, and z-gradient is G_z ¼ ÖB$_z$/Öz.

Ch. 6

- Curie's law and magnetization for protons: $\mathbf{M} \sim \rho\gamma^2(h/2\pi)^2\mathbf{B}/(4kT)$
- Spin excess
- Thermal energy

Curie's law and magnetization

- Boltzmann distribution: Probability = $e^{-\varepsilon/kT}/Z$ where $Z = \Sigma\, e^{-\varepsilon/kT}$
- Magnetization = $\rho\ \Sigma$ (Prob) μ_z where $\mu_z = m\gamma(h/2\pi)$, $\varepsilon = -m\omega(h/2\pi)$, and $\rho = N/V$
- Because $|\varepsilon/kT| << 1$, magnetization ~ $\rho s(s+1)\gamma^2(h/2\pi)^2B/(3kT)$
- For proton, the magnetization ~ $\rho\gamma^2(h/2\pi)^2B/(4kT)$
- Curie's law: magnetization ó 1/T

Spin excess

- Spin excess ΔN = difference between the number of spins parallel to and number of spins anti-parallel to the external field.
- ΔN ¼ N(ı) – N(∅) ~ Nu/2 where $u = (h/2\pi)\omega_0/(kT)$