Chapter 9 – 1D imaging, K-space and Gradient Echos

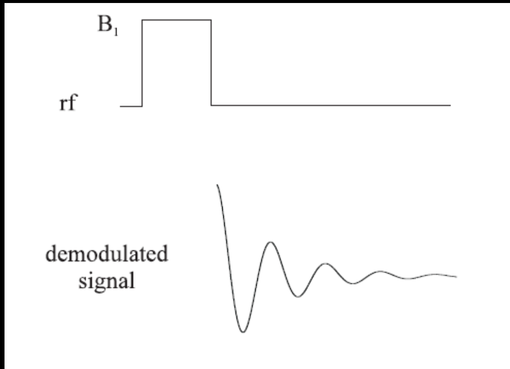
Jaladhar Neelavalli, Ph.D.
Assistant Professor,
WSU SOM - Dept. of Radiology

Quick Recap:

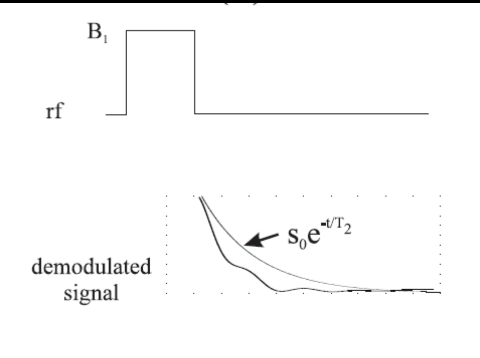
Free Induction Decay

$$s(t) \propto \omega_0 \int d^3r e^{-t/T_2(\vec{r})} \mathcal{B}_\perp(\vec{r}) M_\perp(\vec{r},0) e^{i(\underline{(\Omega - \omega(\vec{r}))t + \phi_0(\vec{r})} - \theta_{\mathcal{B}}(\vec{r}))}$$

$\Omega \neq \omega_0$ Or $\omega \neq \omega_0$



(c)



(d)

$$S(t) = S_0 \int e^{-t/T_2} e^{i(\Omega - \omega(\vec{r}))t} + \phi_0$$

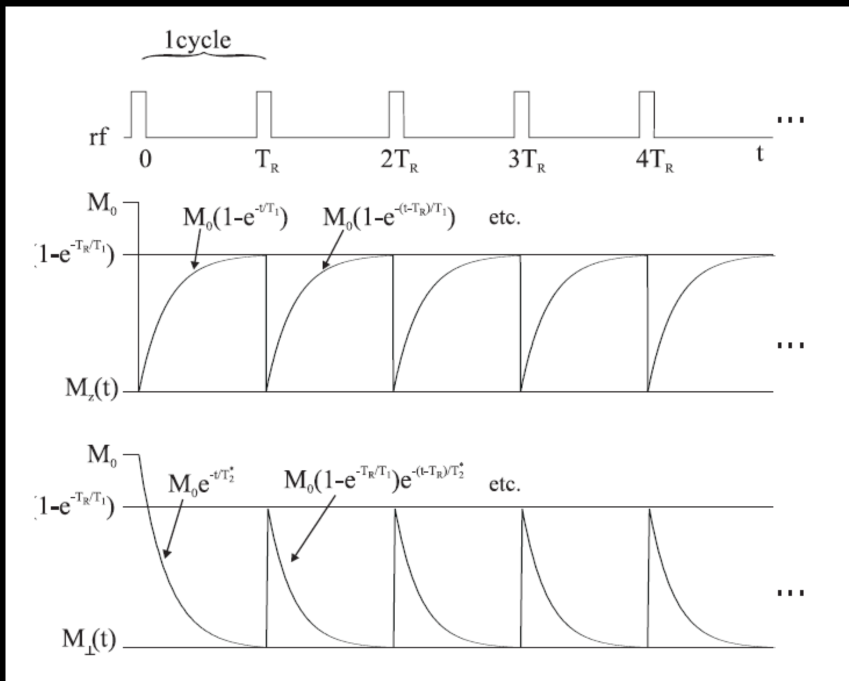
$$e^{-\frac{t}{T_2'}} \sim \int e^{i(\Omega - \omega(\vec{r}))t}$$

$$\boxed{\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}}$$

Faster decay of signal from transverse magnetization occurs due the presence of local field inhomogeneities

Quick Recap:

Multi-pulse experiments



For $TR \gg T_2^*$

$$M_z(t) = M_z(0) \cdot e^{-\frac{t}{T_1}} + M_0 \cdot \left(1 - e^{-\frac{t}{T_1}}\right)$$

$$M_{xy}(t) = M_{xy}(0) \cdot e^{-\frac{t}{T_2^*}}$$

$$M_z(nTR^-) = M_0(1 - e^{-TR/T1})$$

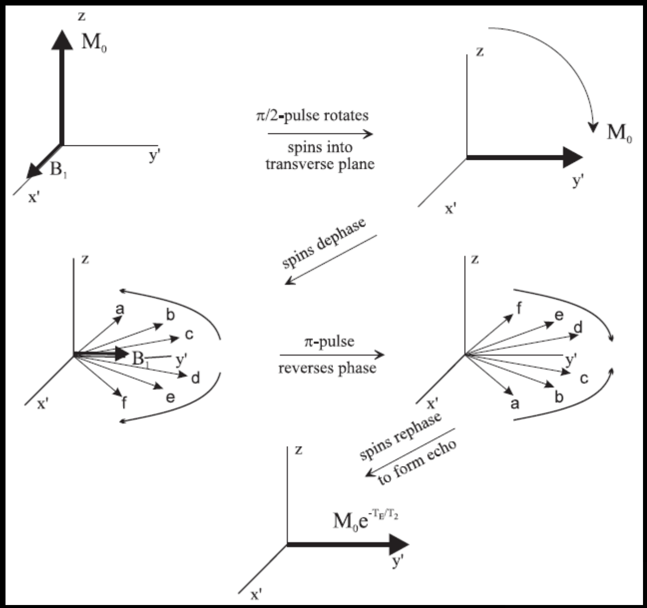
$$M_\perp(nTR^+) = M_0\left(1 - e^{-\frac{TR}{T1}}\right) e^{-\frac{t}{T_2^*}}$$

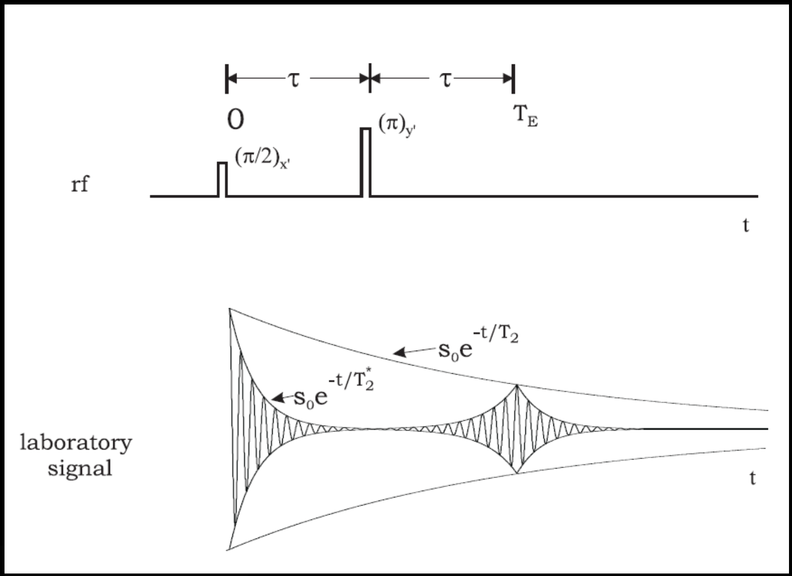
$$M_\perp(nTR^-) = M_0\left(1 - e^{-\frac{TR}{T1}}\right) e^{-\frac{TR}{T_2^*}}$$

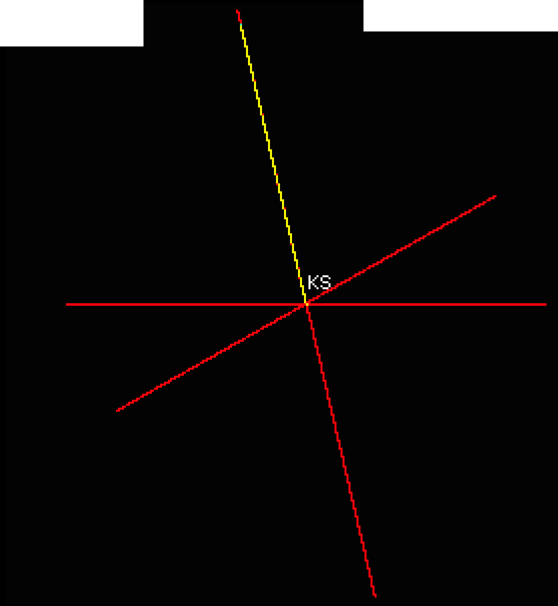
The signal becomes a function of both longitudinal and transverse relaxation times

Quick Recap:

Spin echo (rf echo)







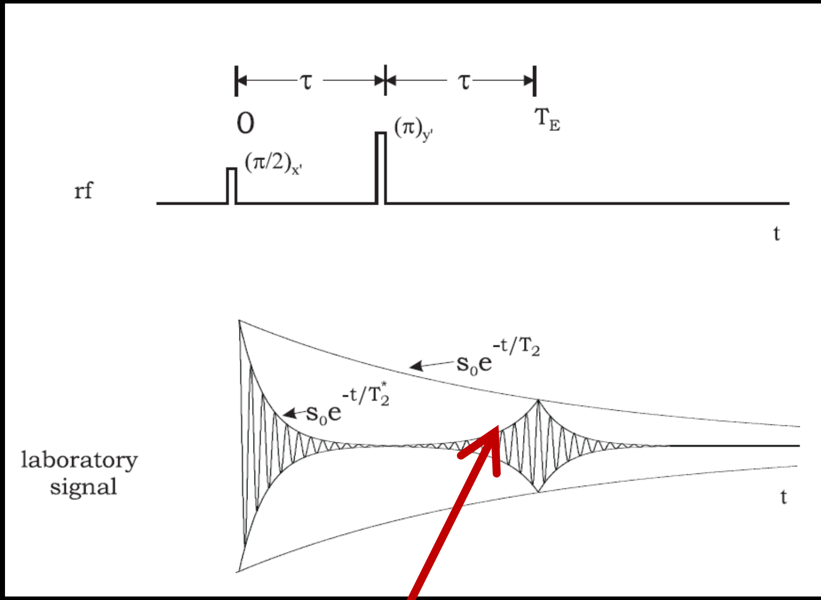
Animation source:
www.kyb.tuebingen.mpg.de/research/dep/ks/spinzoo.html

Quick Recap:

Spin echo (rf echo)

How SE works

1. $\phi(\vec{r}, t) = -\gamma \Delta B(\vec{r}) t, 0 < t < \tau$



Refocusing π pulse after time τ, instantaneous reversal of phase

2. $\phi(\vec{r}, \tau^+) = -\phi(\vec{r}, \tau^-) = \gamma \Delta B(\vec{r}) \tau, t = \tau$

3. $\phi(\vec{r}, t) = -\gamma \Delta B(\vec{r})(t - \tau) + \gamma \Delta B(\vec{r}) \tau \text{ , } \tau < t < \tau$

$$S(t) = S_0 \cdot e^{-t\left(\frac{1}{T_2} - \frac{1}{T_2'}\right)}$$

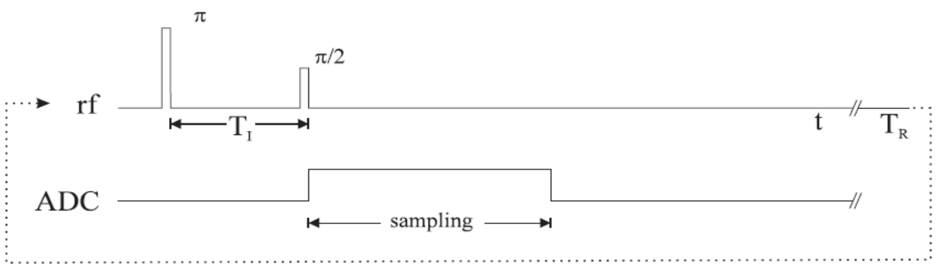
4. $\phi(\vec{r}, 2\tau) = 0, t = 2\tau \equiv TE$

TE – echo time,
Phase of the spins at TE is zero

Quick Recap:

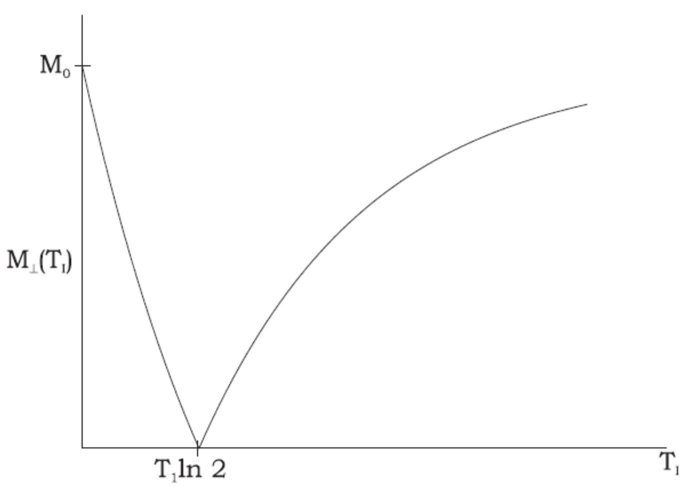
Inversion Recovery

$$M_z(t) = M_z(@_{t=0}) \cdot e^{-\frac{t}{T_1}} + M_0 \cdot \left(1 - e^{-\frac{t}{T_1}}\right)$$



(a)

$$T_{I,null} = T_1 \ln 2$$



$$M_z(0^+) \quad = \quad -M_0$$

$$M_z(t) = -M_0 e^{-t/T_1} + M_0(1 - e^{-t/T_1}) = M_0(1 - 2e^{-t/T_1})$$

$$0 < t < T_I$$

$$M_\perp(t) = \left| M_0(1 - 2e^{-T_I/T_1}) \right| e^{-(t-T_I)/T_2^*}, \qquad t > T_I$$

1D imaging, k-space and gradient echos

- Effective Spin density
- Frequency encoding
- k-space representation
- Gradient Echo

Signal equation and effective spin density

$$s(t) \propto \omega_0 \int d^3r e^{-t/T_2(\vec{r})} \mathcal{B}_\perp(\vec{r}) M_\perp(\vec{r},0) e^{i((\Omega-\omega(\vec{r}))t+\phi_0(\vec{r})-\theta_{\mathcal{B}}(\vec{r}))}$$

Ignoring relaxation effects and assuming uniform coil reception and excitation profiles…

$$S(t) = \Lambda \cdot \omega_0 \cdot \beta_\perp \int d^3r \cdot M_\perp(\vec{r},0) \cdot e^{i(\Omega-\omega(\vec{r},t))t}$$

Λ is the term that includes the electronic gain terms

Magnitude of M_0 (for ^1H),

$$M_0(\vec{r}) = \rho_0(\vec{r}) \frac{\gamma^2 \cdot \hbar^2 \cdot B_0}{4 \cdot k \cdot T}$$

ρ_0 is the # of spins per unit volume

Signal equation and effective spin density

$$S(t) = \Lambda \cdot \omega_0 \cdot \beta_\perp \int d^3r \cdot M_\perp(\vec{r},0) \cdot e^{i(\Omega-\omega(\vec{r},t))t}$$

$$S(t) = \int d^3r \cdot \rho(\vec{r}) \cdot e^{i(\Omega-\omega(\vec{r},t))t}$$

where

$$\rho(\vec{r}) = \Lambda \cdot \omega_0 \cdot \beta_\perp \cdot M_0(\vec{r}) = \Lambda \cdot \omega_0 \cdot \beta_\perp \cdot \rho_0(\vec{r}) \frac{\gamma^2 \cdot \hbar^2 \cdot B_0}{4 \cdot k \cdot T}$$

… the effective spin density

Note: We ignored relaxation affects in this.
Practically for repeat acquisitions where the signal depends both on T1 and T2, there is some weighting of T1 and T2 in the effective spin density.

The phase term

$$S(t) = \int d^3r \cdot \rho(\vec{r}) \cdot e^{i(\Omega - \omega(\vec{r},t))t}$$

Phase term…

$$\phi(\vec{r},t) = \int_0^t \left(\Omega - \omega(\vec{r},t')\right) \cdot dt'$$

Demodulated signal…

$\Omega = \omega_0,$ $\qquad \omega(\vec{r},t) = \omega_0 + \Delta\omega(\vec{r},t)$

$$\boxed{\phi(\vec{r},t) = \int_0^t -\Delta\omega(\vec{r},t') \cdot dt'}$$

$$\phi(\vec{r},t) = -\Delta\omega(\vec{r}) \cdot t$$

$$\boxed{S(t) = \int d^3r \cdot \rho(\vec{r}) \cdot e^{i\phi(\vec{r},t)}}$$

Encoding spatial information: Imaging

The goal is to differentiate signals coming from different spatial locations

In the early 1970s both Paul Lauterbur and Peter Mansfield realized that by forcing the B_0 field to vary on purpose, one could recover such spatial information from the object

1D Imaging

Their approach was that by applying a magnetic field gradient in one direction one could spatially distinguish spins along that direction in terms of their precession frequency, i.e., if

$$B(z) = B_0 + G \cdot z$$

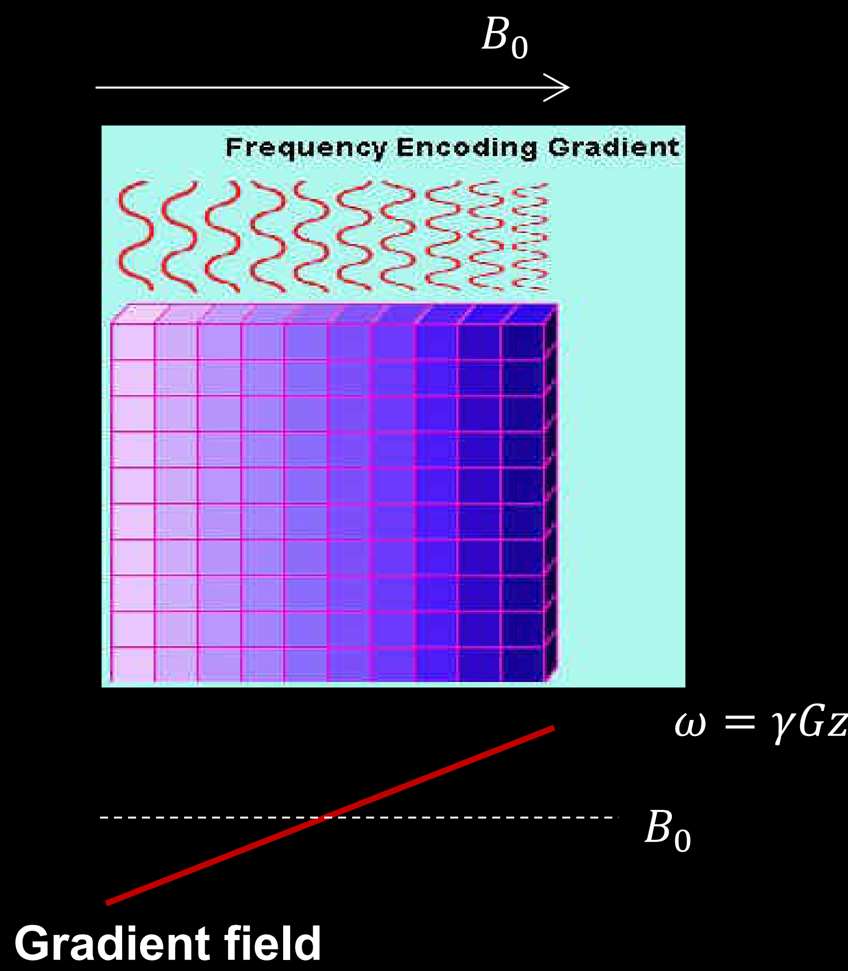
$$\boxed{\omega(z) = \omega_0 + \gamma \cdot G \cdot z}$$

… i.e., for every position z in the object there is a unique frequency $\omega(z)$

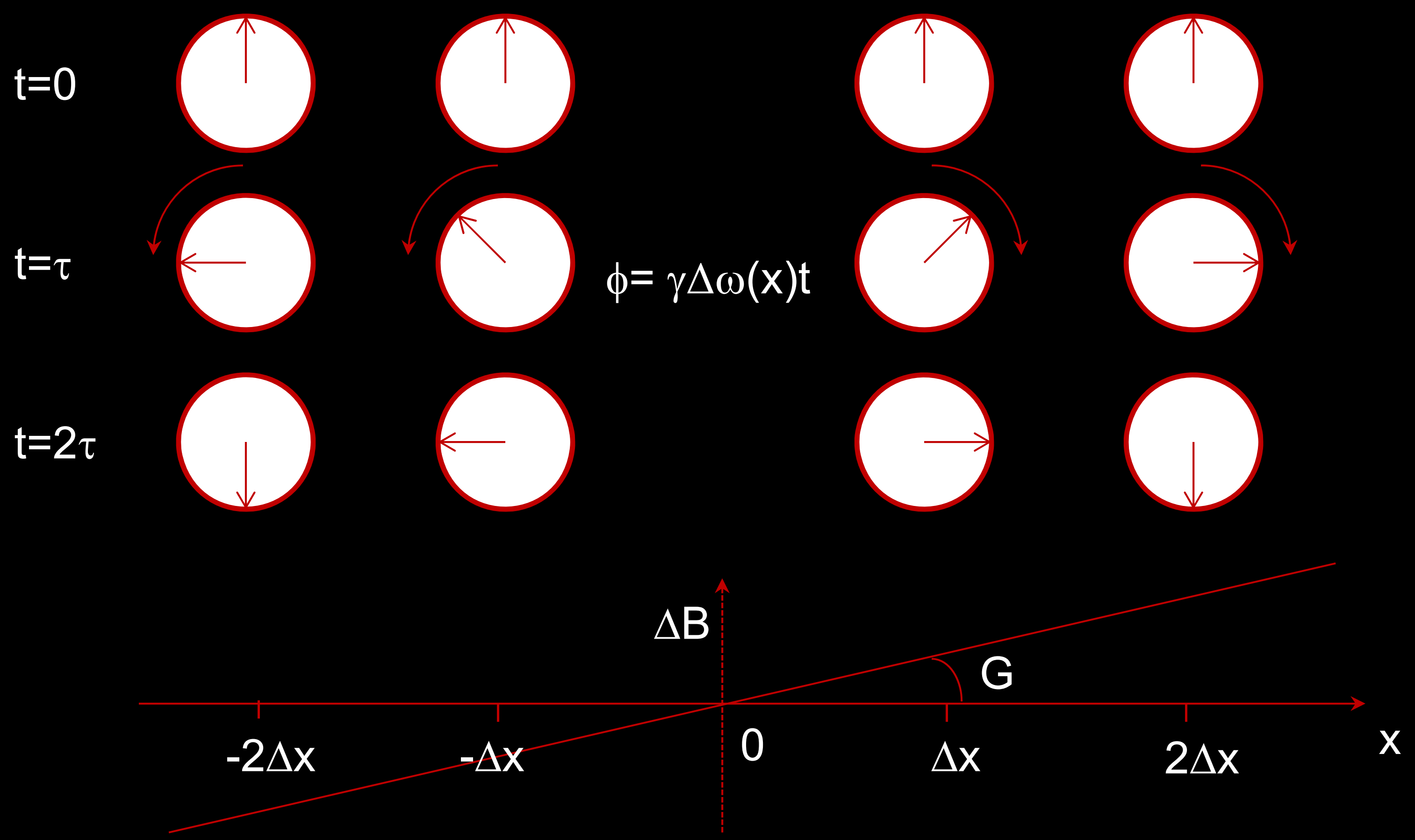
1D Imaging

$$B(z) = B_0 + G \cdot z$$

$$\boxed{\omega(z) = \omega_0 + \gamma \cdot G \cdot z}$$



Frequency encoding



1D Imaging

$$\boxed{S(t) = \int d^3r \cdot \rho(\vec{r}) \cdot e^{i\phi(\vec{r},t)}}$$

$$\phi(\vec{r},t) = \int_0^t \left(\Omega - \omega(\vec{r},t')\right) \cdot dt'$$

$$\Omega = \omega_0 \qquad \omega(\vec{r},t) = \omega_0 + \Delta\omega(\vec{r},t) \qquad \omega(z) = \omega_0 + \gamma \cdot G \cdot z$$

$$\varphi(\vec{r},t) = \int_0^t -\Delta\omega(\vec{r},t') \cdot dt' \qquad \phi(z,t) = \int_0^t -\gamma \cdot G \cdot z \cdot dt'$$

$$\Rightarrow \phi(z,t) = -\gamma \cdot G \cdot z \cdot t$$

$$\boxed{S(t) = \int dz \cdot \rho(z) \cdot e^{-i\cdot\gamma\cdot G\cdot z\cdot t}}$$

1D Imaging

$$S(t) = \int dz \cdot \rho(z) \cdot e^{-i \cdot \gamma \cdot G \cdot z \cdot t}$$

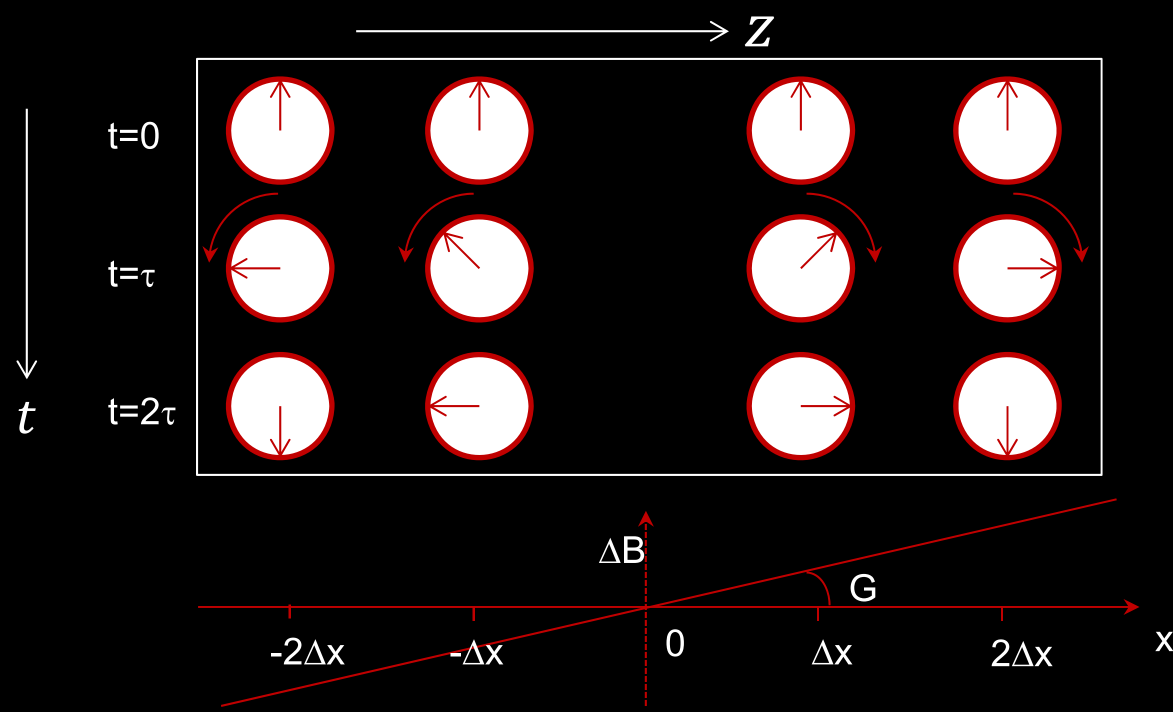
$$\phi(z,t) = 2\pi \cdot k \cdot t \qquad k = \frac{\gamma}{2\pi} \cdot G \cdot t \qquad k = \bar{\gamma} \cdot G \cdot t \qquad \bar{\gamma} \equiv \gamma$$

$$\boxed{S(t) = S(k) = \int dz \cdot \rho(z) \cdot e^{-i \cdot 2\pi \cdot k \cdot z}}$$

… the 1D imaging equation

Signal is the Fourier Integral in spatial frequency variable k

Frequency encoding



$$s(t) = \int dz \cdot \rho(z) \cdot e^{i \cdot \phi(z,t)} \longrightarrow s(t) = \int dz \cdot \rho(z) \cdot e^{-i \cdot \gamma \cdot G \cdot z \cdot t}$$

$s(t)$ contains signal contributions from all excited spins in the sample

1D Imaging

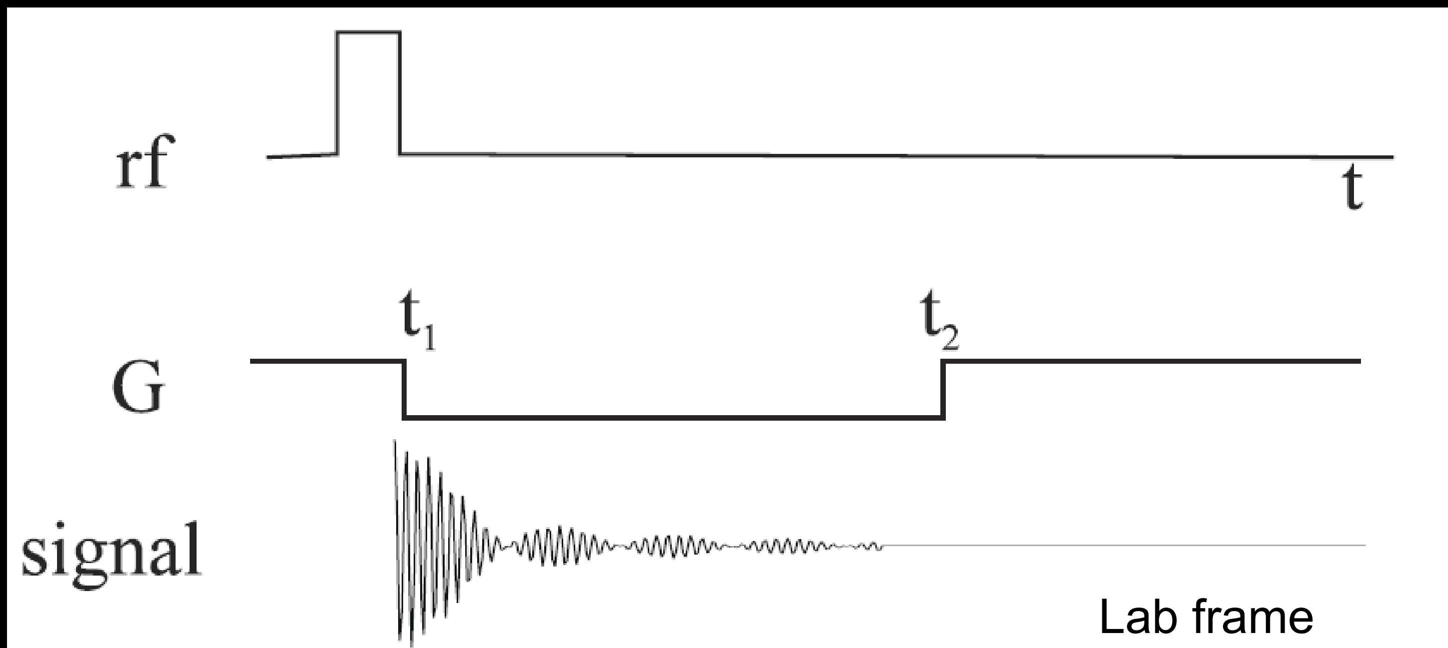
Fourier Transform Pair

$$S(t) = S(k) = \int_{-\infty}^{+\infty} dz \cdot \rho(z) \cdot e^{-i \cdot 2\pi \cdot k \cdot z}$$

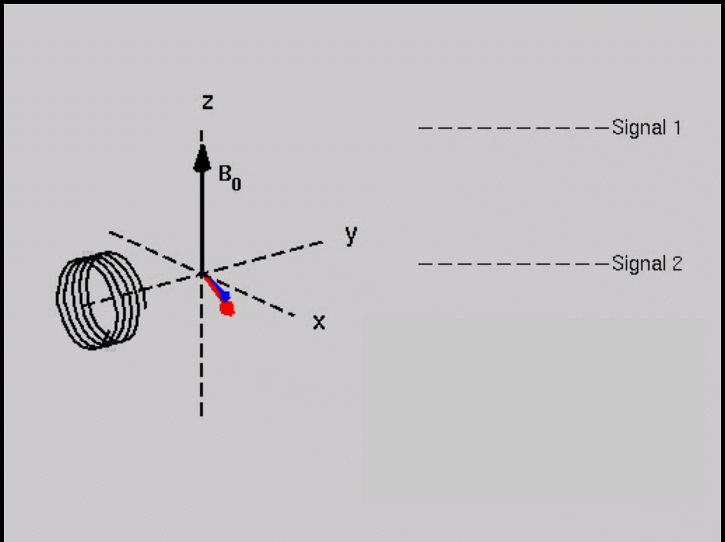
Inverse Fourier Transform…

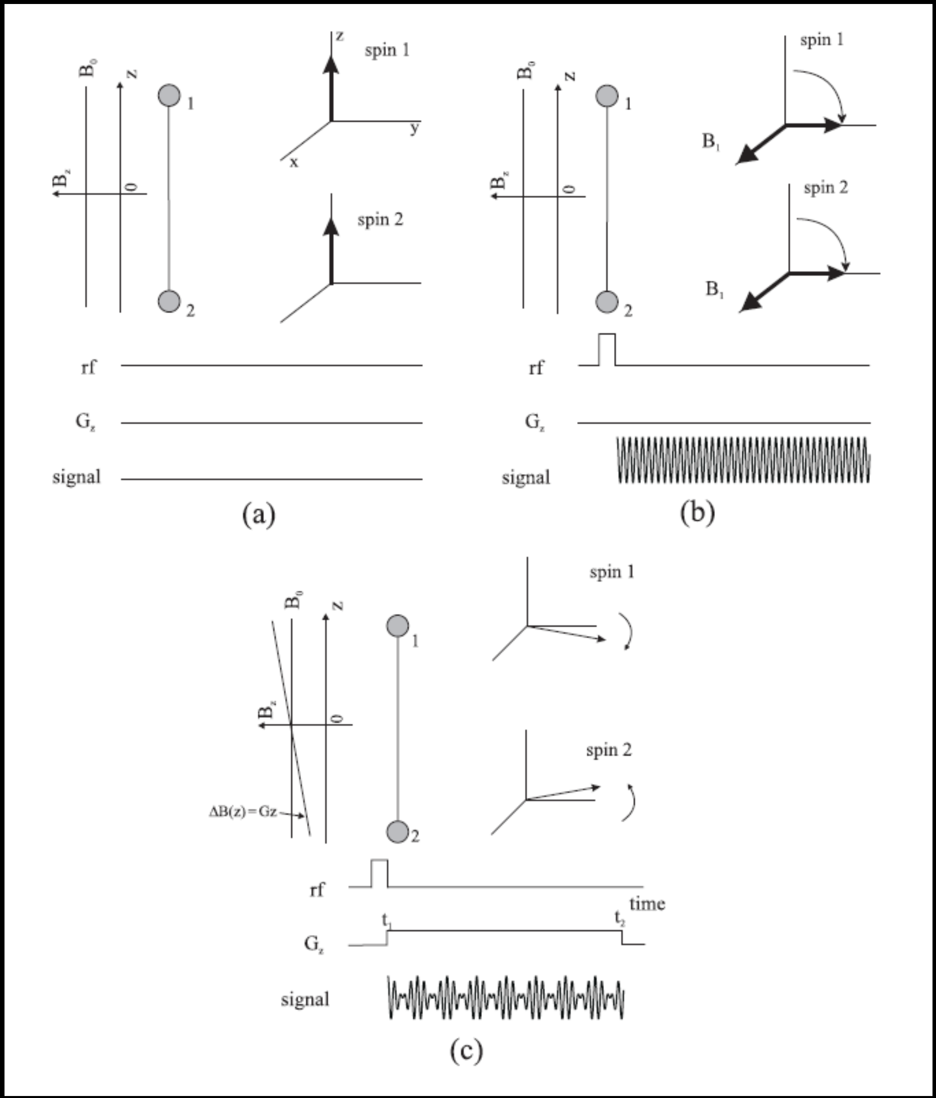
$$\rho(z) = \int_{-\infty}^{+\infty} dk \cdot S(k) \cdot e^{+i \cdot 2\pi \cdot k \cdot z}$$

Gradient ON



A simple two spin system





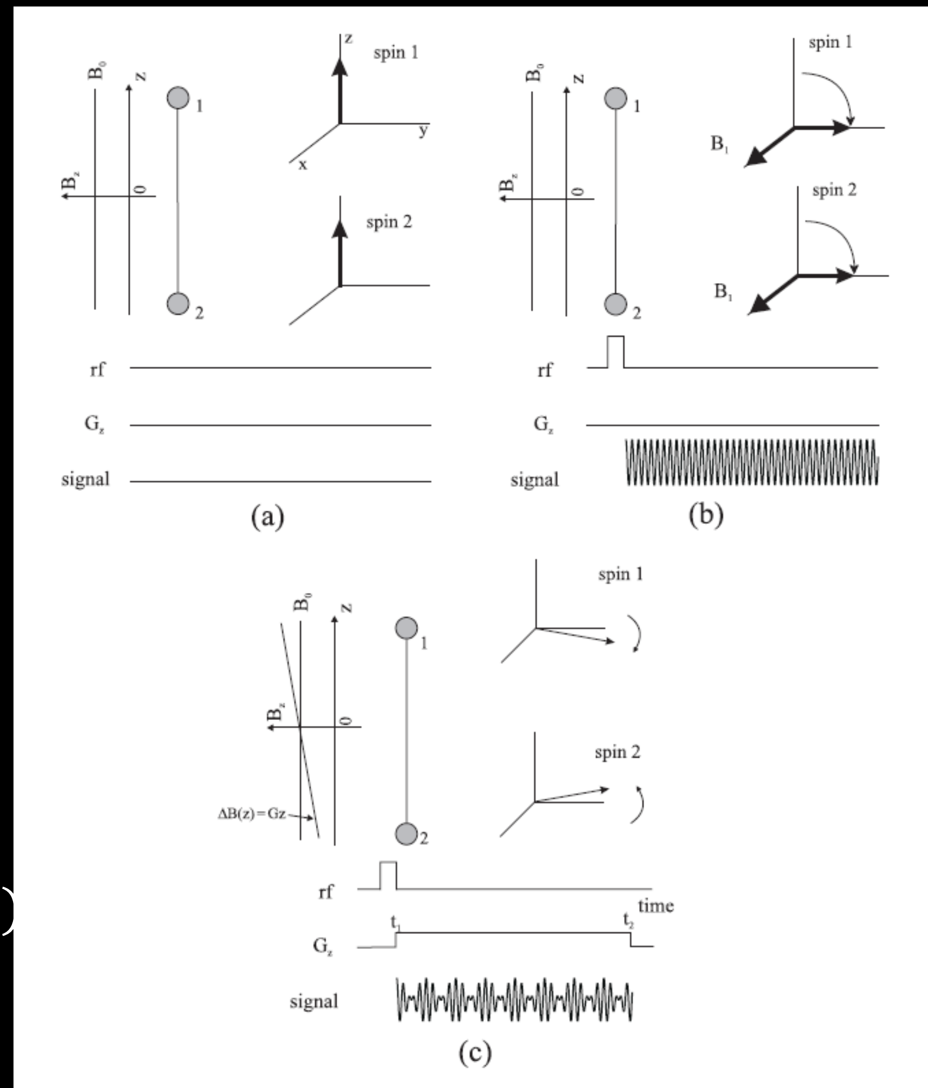
A simple two spin system

$$s(t) = S_0\left(e^{-i\gamma G z_0 t} + e^{i\gamma G z_0 t}\right)$$
$$= 2S_0\cos(\gamma G z_0 t)$$

$$s(k) = 2S_0\cos(2\pi k z_0)$$

$$\Downarrow$$

$$\rho(z) = \int_{-\infty}^{\infty} dk 2S_0\cos(2\pi k z_0) e^{i2\pi k z}$$
$$= S_0 \int_{-\infty}^{\infty} dk \left(e^{i2\pi k(z+z_0)} + e^{i2\pi k(z-z_0)}\right)$$
$$= S_0[\delta(z+z_0) + \delta(z-z_0)]$$



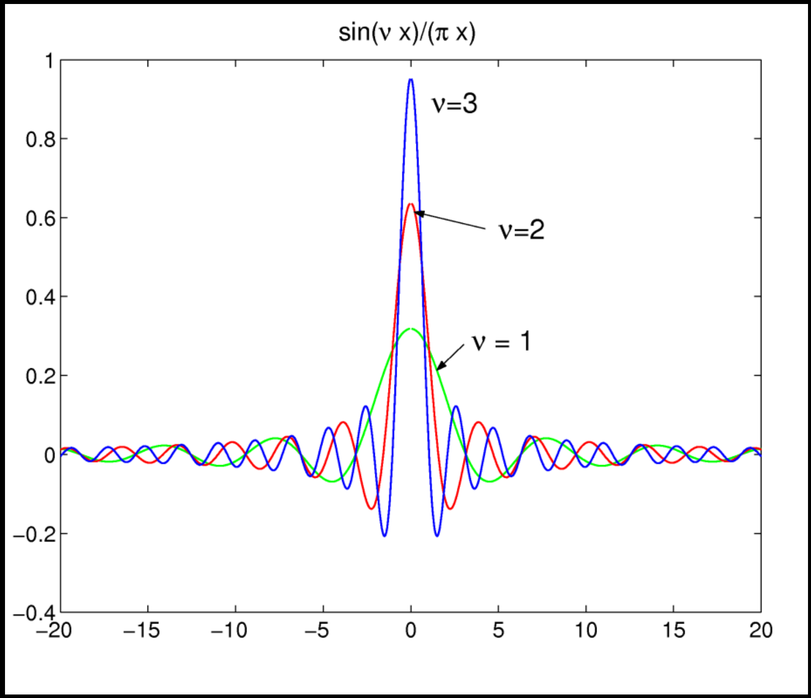
Dirac Delta Function

$$\begin{cases} \delta(z-a) = 0, & z \neq a \\ \int_{a-\varepsilon}^{a+\varepsilon} dz\delta(z-a) = 1, & \varepsilon \to 0 \end{cases}$$

$$\delta(z-a) = \int_{-\infty}^{\infty} dk\, e^{i2\pi k(z-a)}$$

$$FT[\delta(z-a)] = e^{-i2\pi ka}$$

$$\delta(z-a) = IFT[e^{-i2\pi ka}]$$

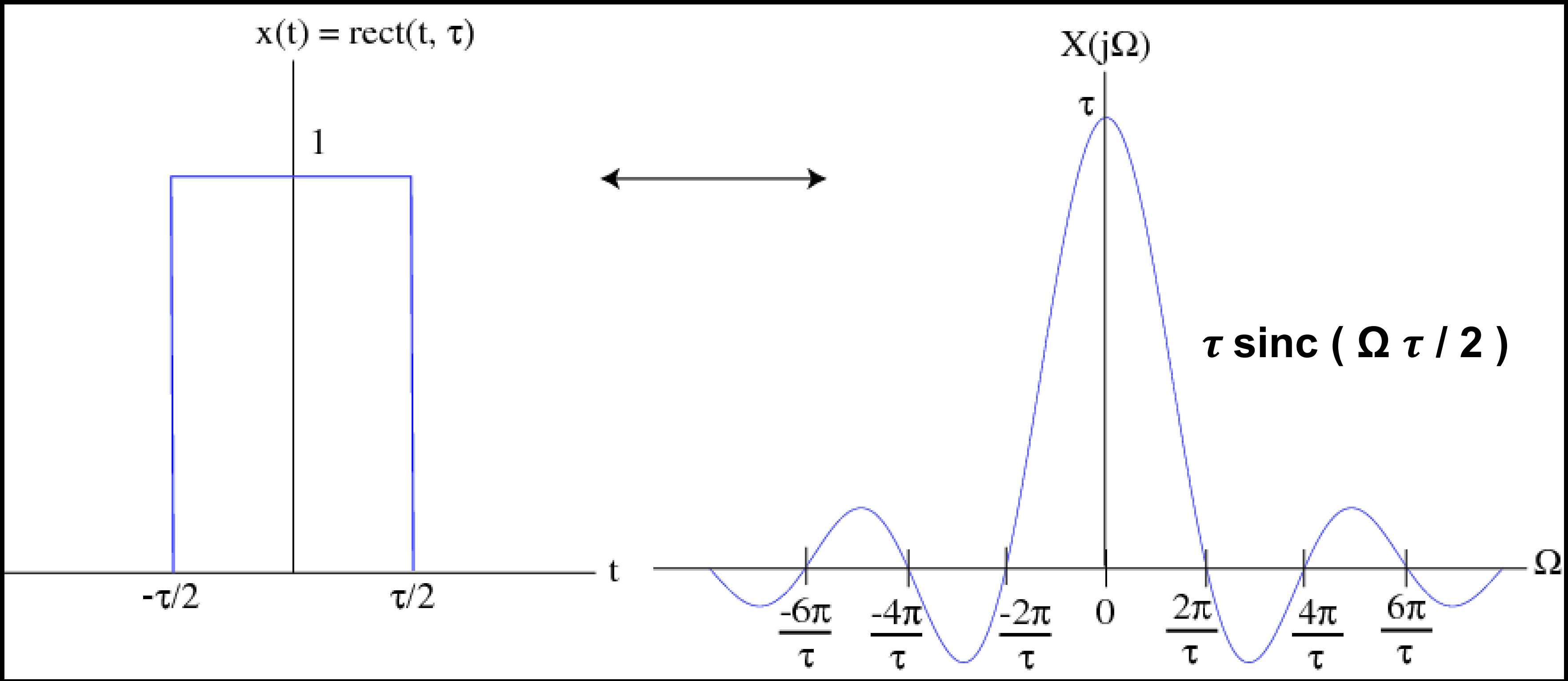


$$\int_{-\infty}^{\infty} dz\delta(z-a)f(z) = f(a)$$

The delta function helps us sample the function's value at a particular point

Image Source:
http://upload.wikimedia.org/wikipedia/commons/4/46/Dirac_delta_sinc.png

Sinc and the rect functions

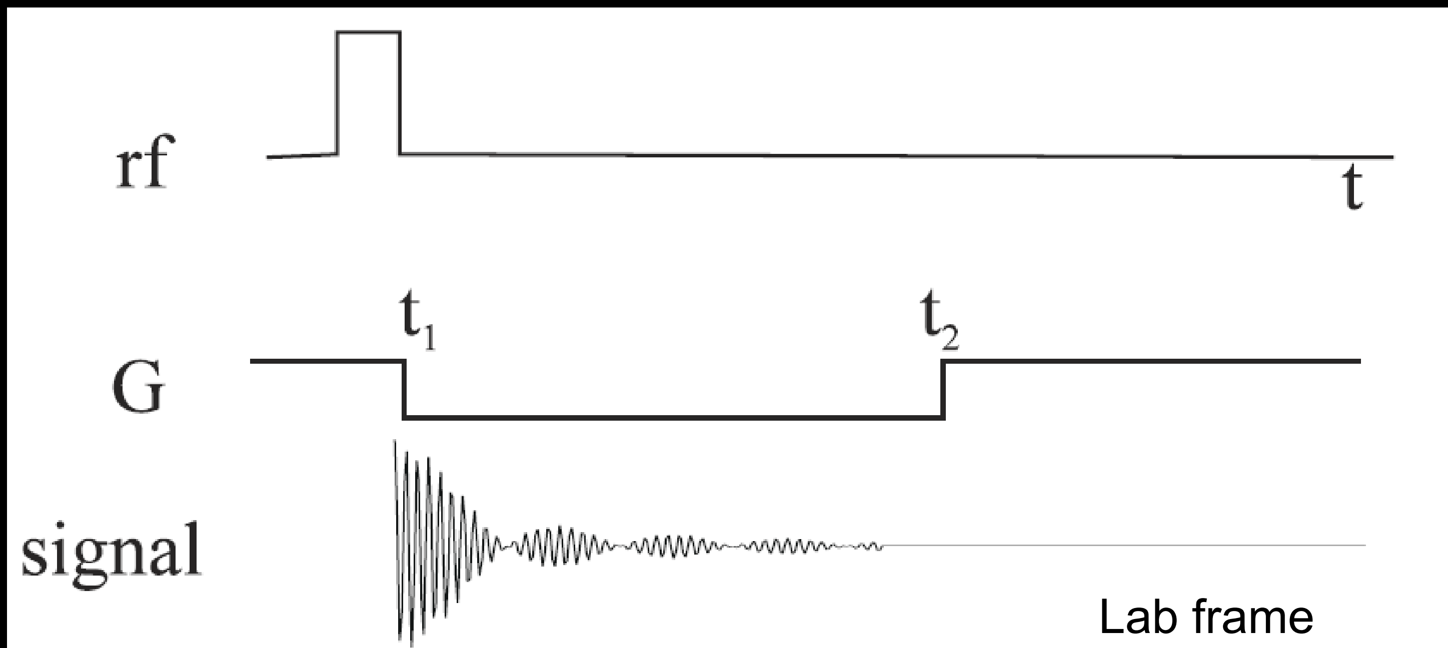


rect $(\tau) \equiv 1$, t ≤ | τ / 2 |
0 , t > | τ / 2 |

A Fourier Pair

Image Source:
http://cnx.org/contents/7549bfa0-d833-4795-a522-60269735215a@8

Gradient ON



k-Space

Spatial frequency variable - k

$$\phi(z,t) = \int_0^t -\gamma \cdot G(t) \cdot z \cdot dt' = 2\pi \cdot z \int_0^t -\bar{\gamma} \cdot G(t) \cdot dt'$$

$$\boxed{k(t) = \int_0^t \bar{\gamma} \cdot G(t) \cdot dt'}$$

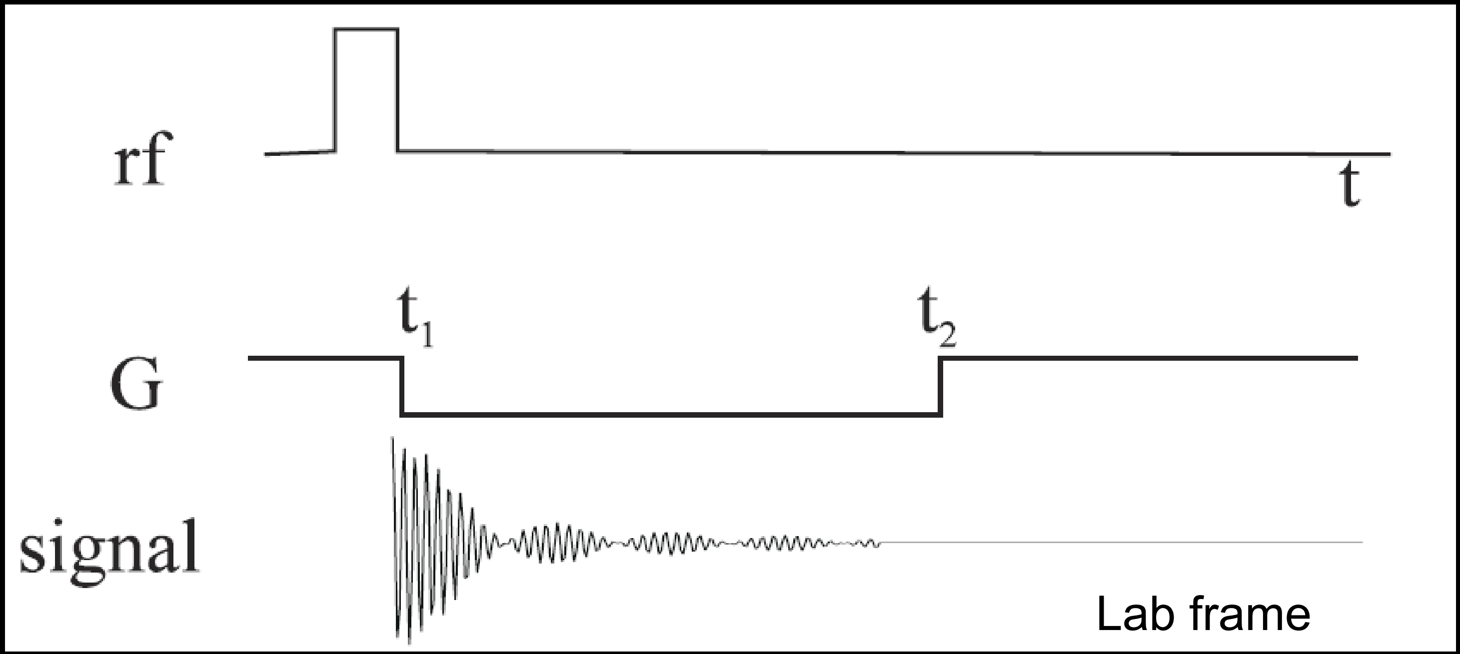
$$\phi(z,t) = -2\pi \cdot k(t) \cdot z$$

$$S(t) = S(k) = \int_{-\infty}^{+\infty} dz \cdot \rho(z) \cdot e^{-i \cdot 2\pi \cdot k(t) \cdot z}$$

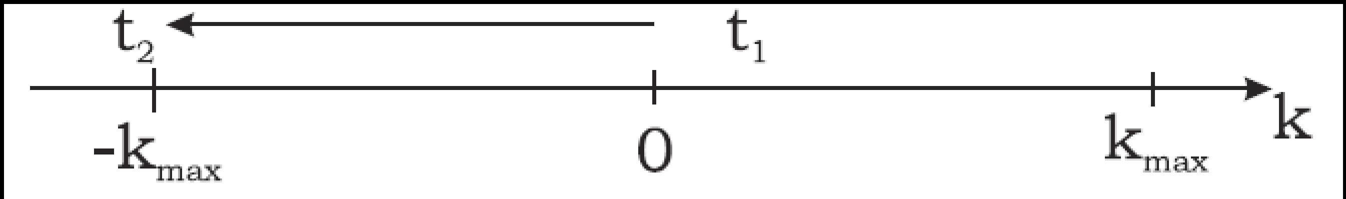
… k takes both positive and negative values

k – Space coverage

Gradient ON

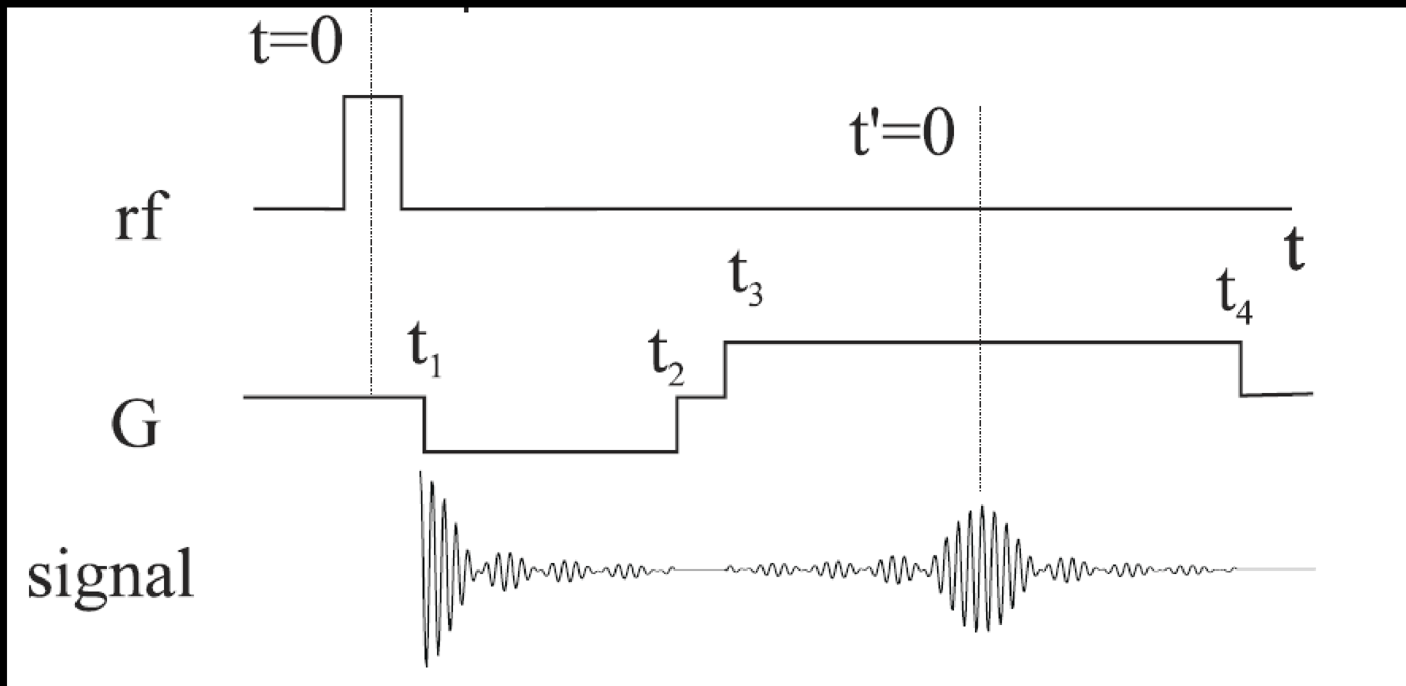


$$k = \bar{\gamma} \cdot -G \cdot t \qquad t_1 < t < t_2$$

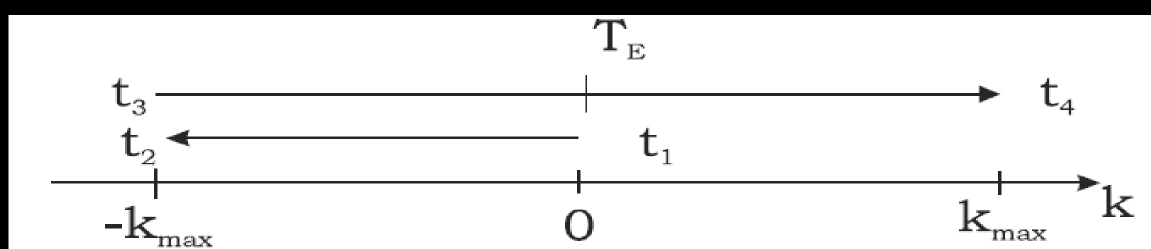


k – Space coverage

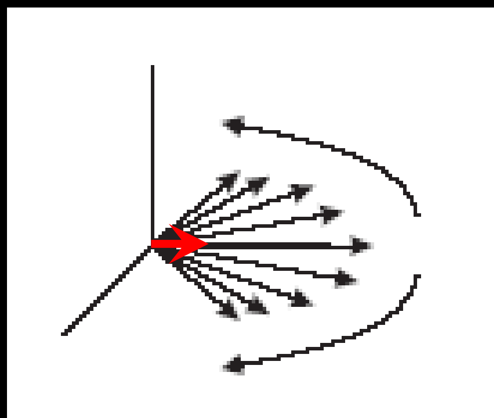
Gradient ECHO



$k = \bar{\gamma} \cdot G \cdot t$ $t_3 < t < t_4$



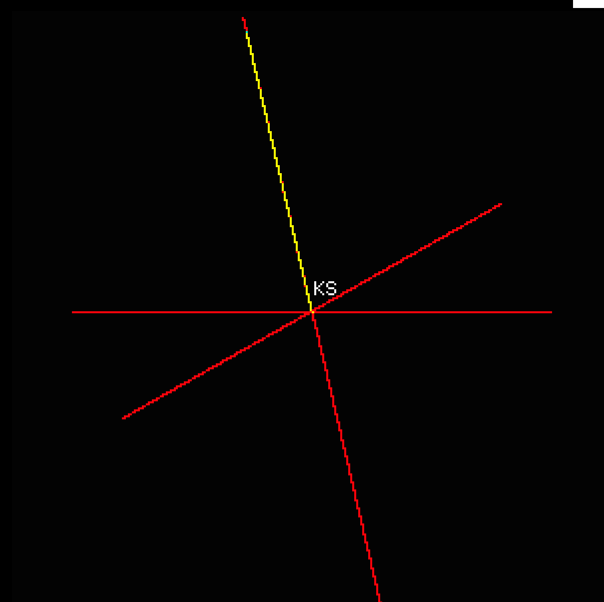
Gradient echo …

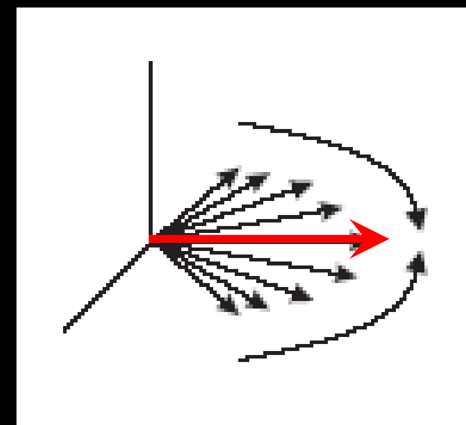


B_o

$-G_x x$

Dephase





B_o

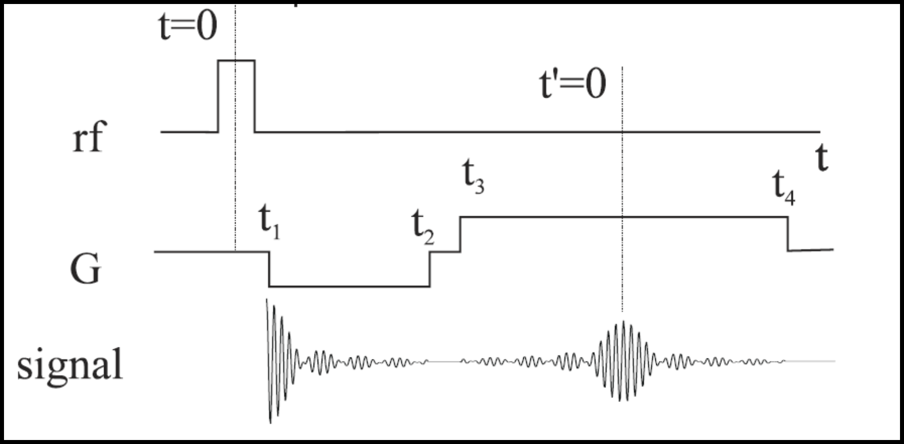
$+G_x x$

Rephase

Animation source:
www.kyb.tuebingen.mpg.de/research/dep/ks/spinzoo.html

Gradient echo …

$$s(t) = \int dz \cdot \rho(z) \cdot e^{i \cdot \phi_G(z,t)}$$



$$\phi_G(z,t) = +\gamma G z (t - t_1) \qquad t_1 < t < t_2$$

$$\phi_G(z,t) = +\gamma G z (t_2 - t_1) - \gamma G z (t - t_3) \qquad t_3 < t < t_4$$

$$t = t_3 + t_2 - t_1 \equiv T_E$$

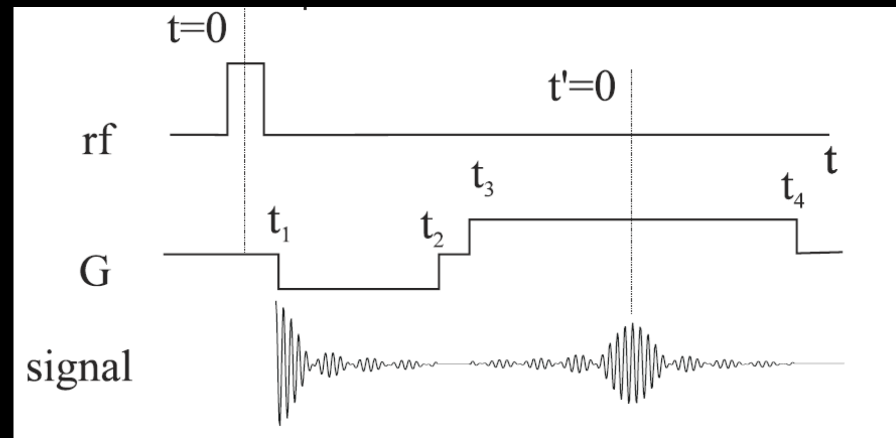
For the gradient echo to occur, the first moment of the gradient waveform should be zero at TE

$$\int G(t) dt = 0$$

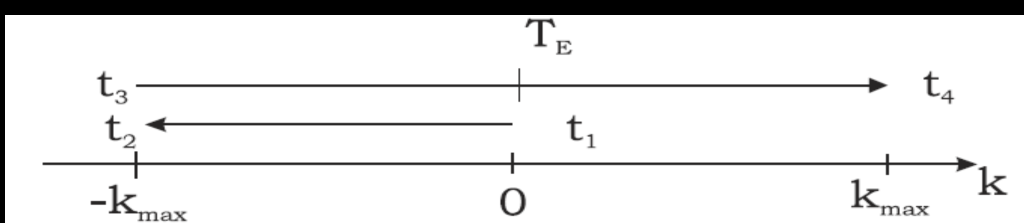
Gradient echo …

Shifting the time origin to coincide with the echo time TE

$$t' \equiv t - t_3 - (t_2 - t_1) = t - T_E$$

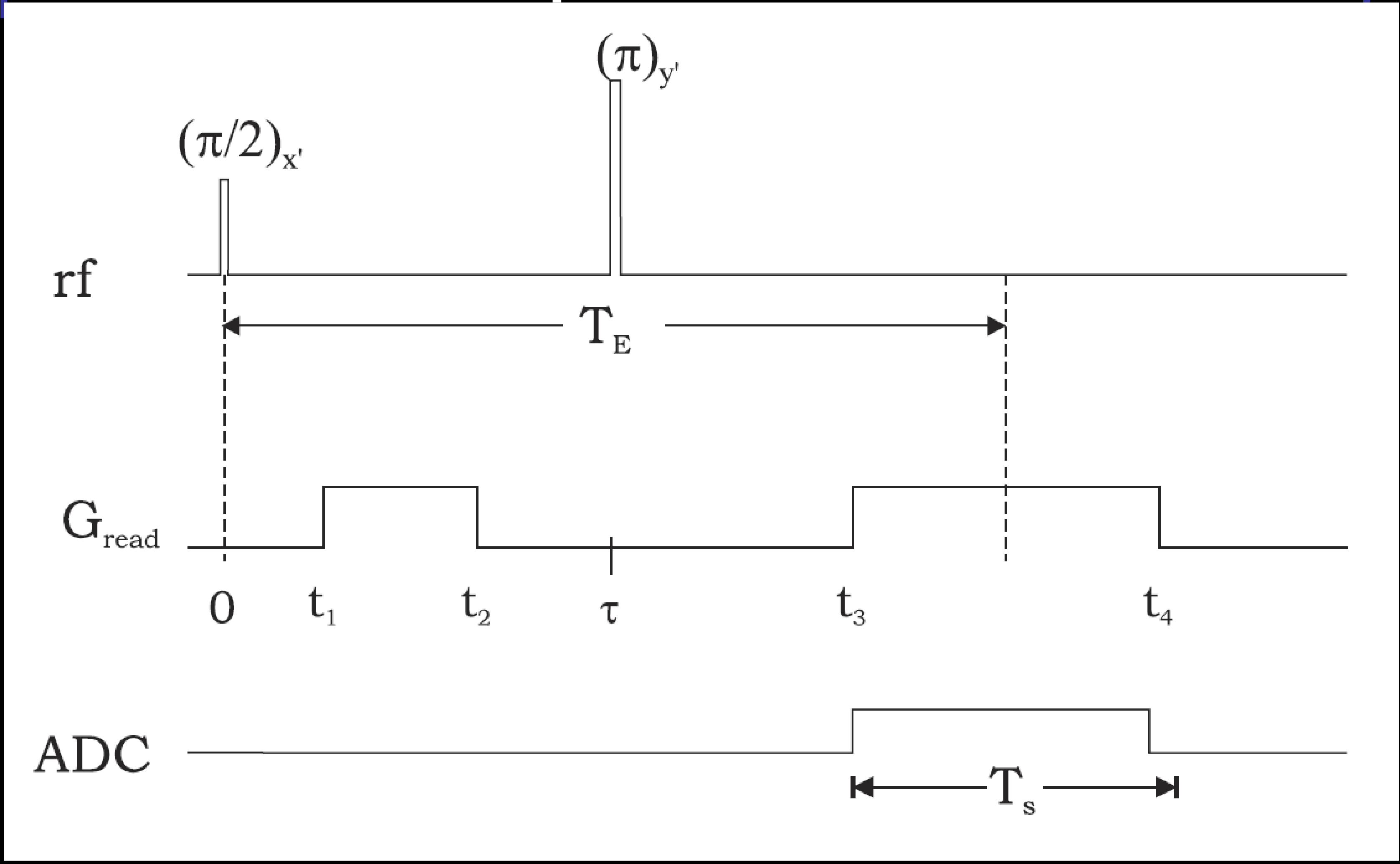


$$\phi_G(z,t) = -\gamma G z t' \qquad -(t_4 - t_3)/2 < t' < (t_4 - t_3)/2$$



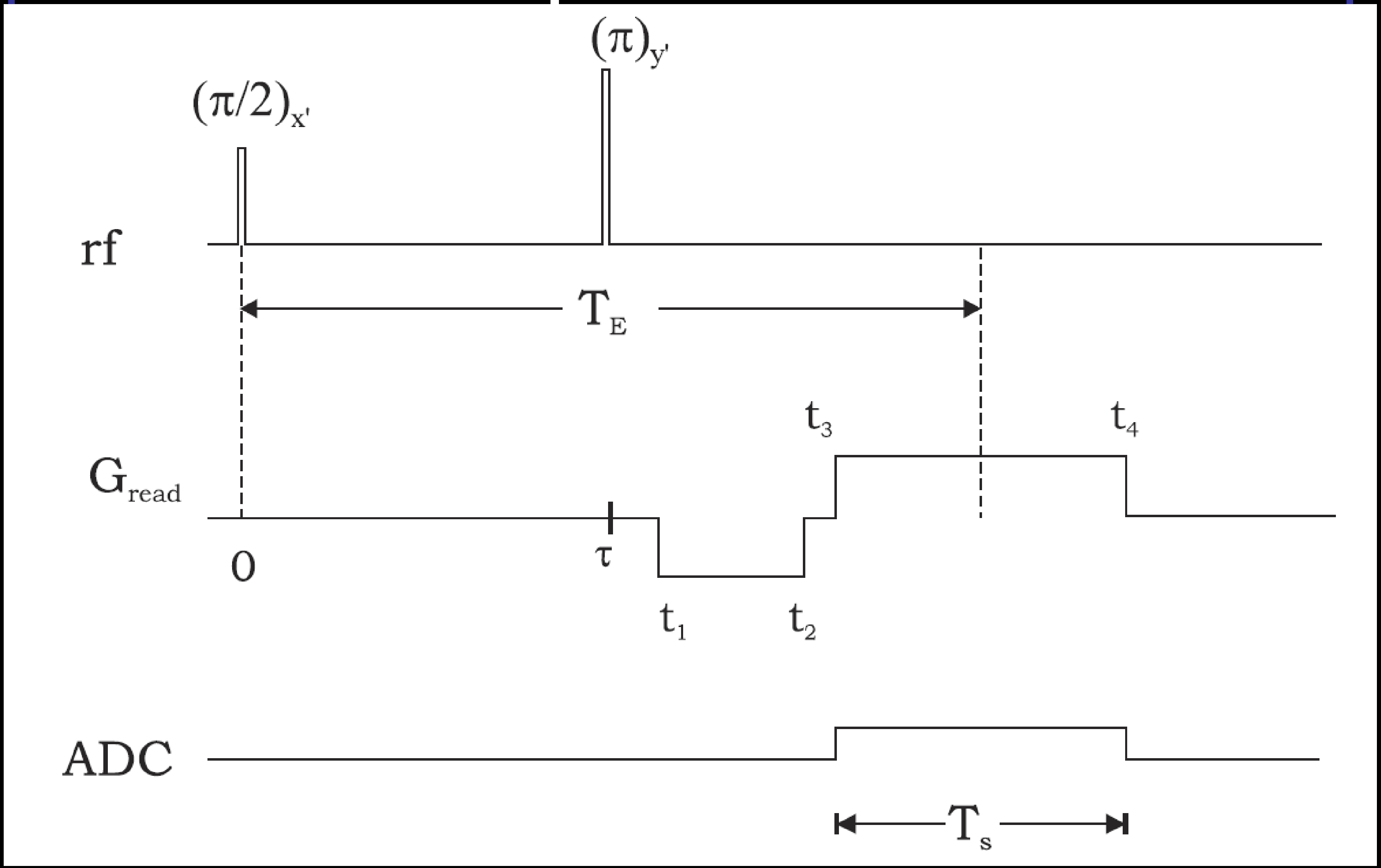
$$\begin{aligned} s(t') &= \int dz \rho(z) e^{-i\gamma G z t'} \\ &= \int dz \rho(z) e^{-i2\pi k(t')z} \end{aligned}$$

Spin echo 1D imaging experiment



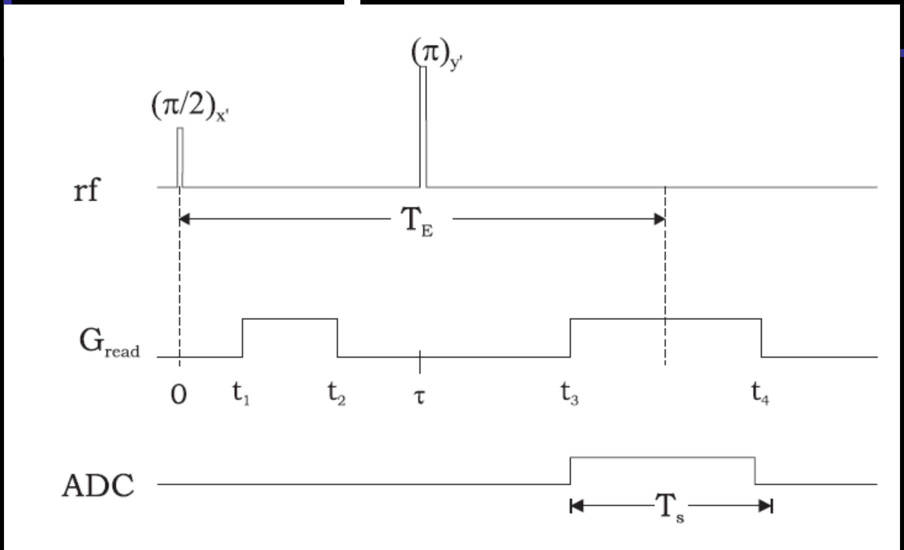
Case - 1

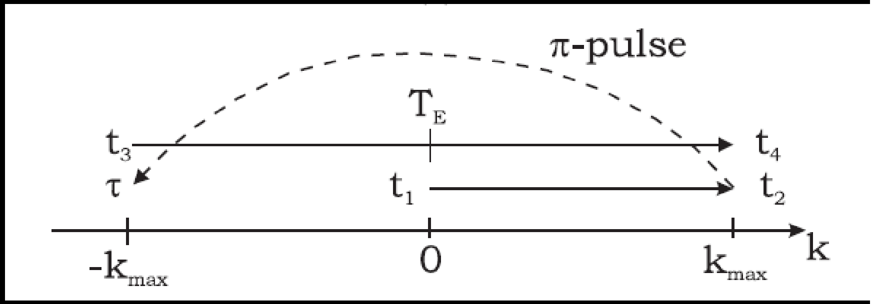
Spin echo 1D imaging experiment

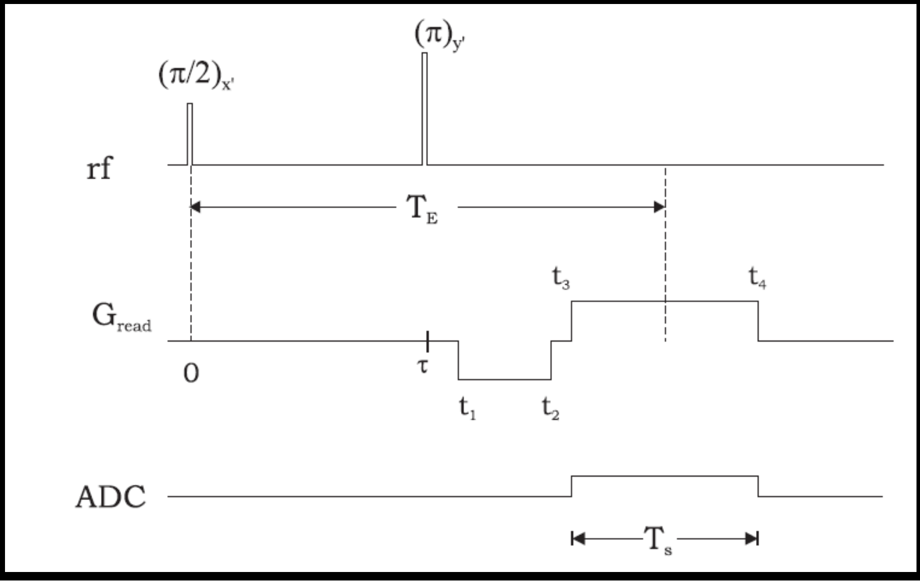


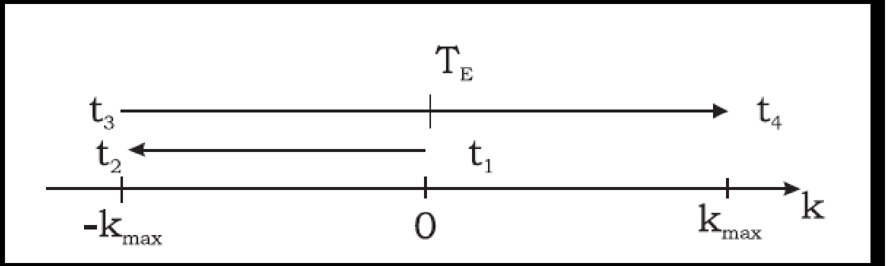
Case - 2

Spin Echo 1D k-space





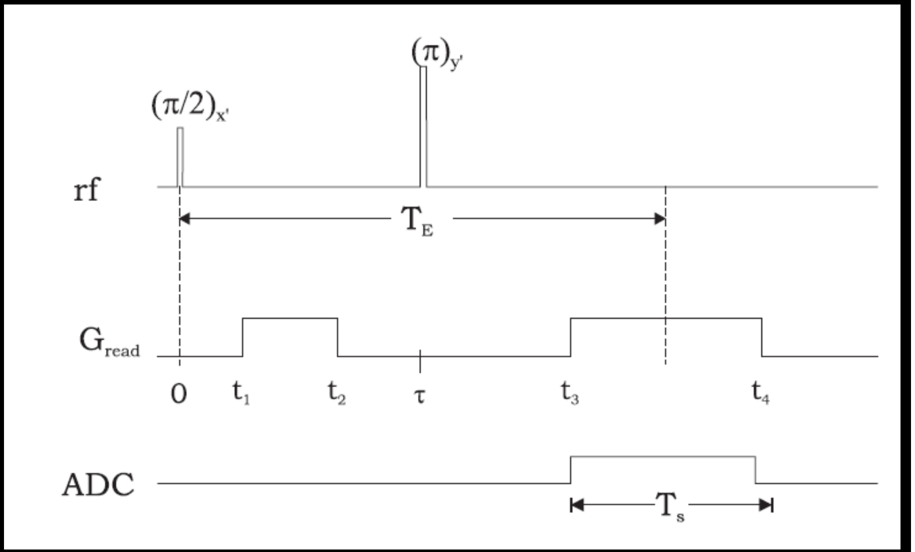




Spin Echo 1D k-space

$$\phi(z,t) = -\gamma \Delta B(z) t - \gamma G z (t - t_1)$$

$$t_1 < t < t_2$$



$$\phi(z,t) = \gamma \underline{\Delta B(z)\tau} + \gamma G z (t_2 - t_1) - \gamma \underline{\Delta B(z)(t - \tau)} - \gamma G z (t - t_3) \quad t_3 < t < t_4$$

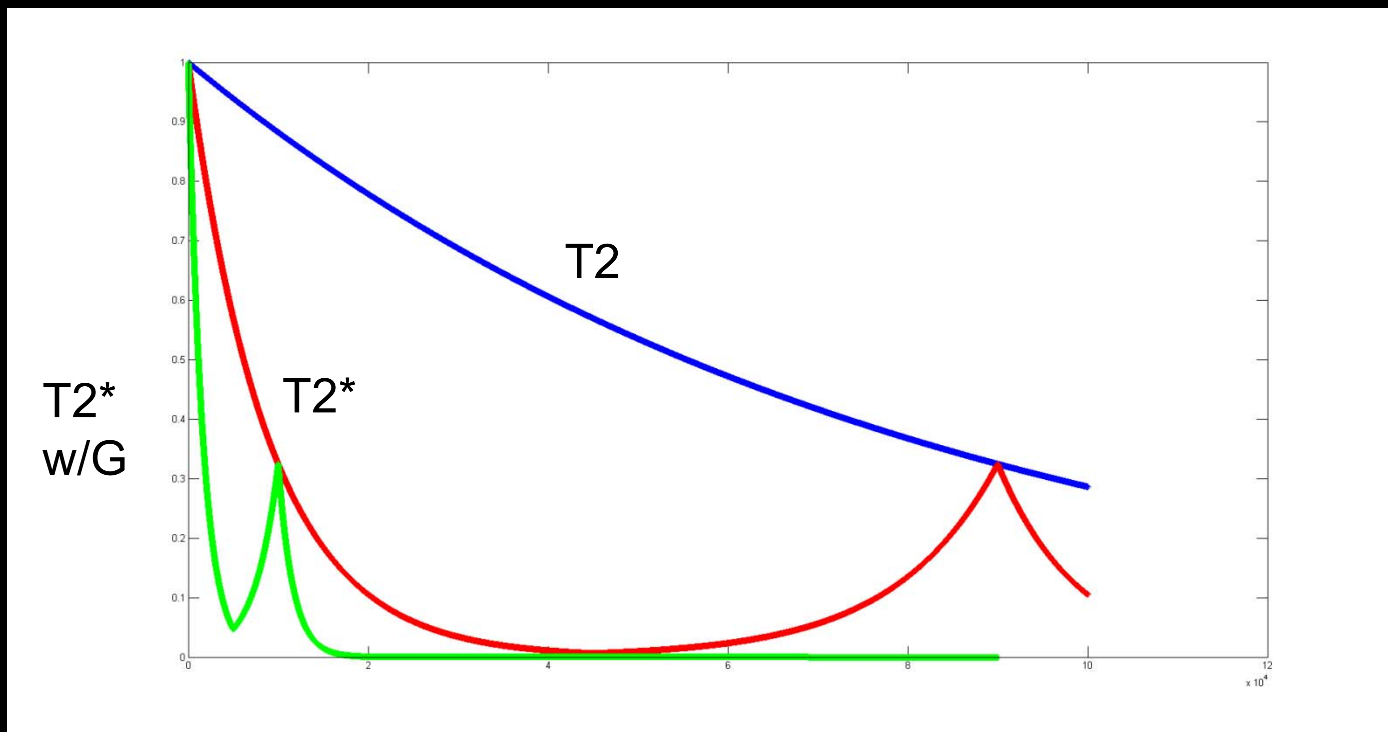
SE

$TE_{SE} = 2\tau$

GRE

$TE_{GRE} = t_3 + (t_2 - t_1)$

GRE vs SE imaging experiment

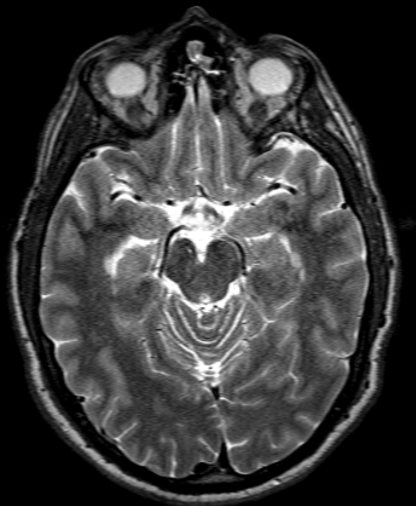


- Faster Imaging is possible with GRE
- Low RF pulse energy
- Does not suffer from imperfect π (180^o) pulses.

SE and T_2

- Why do we need SE?
  - Introduce T2 weightings
  - Reduce signal loss due to field inhomogeneity

GRE, T1W+T2*W



SE, T2W

Homework

- Prob. 9.1, 9.2, 9.4, 9.5

Next Class

Chapter10 - sections 10.1-10.4